Using game theory for the electricity market

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N° 5274

April 2005

Thème NUM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--5563--FR+ENG

Using game theory for the electricity market

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Thème NUM — Systèmes numériques
Projets Omega et Coprin

Rapport de recherche n° 5274 — April 2005 — 43 pages

Abstract: We consider a static game model to describe a spot electricity market in which $\mathcal{S}$ suppliers offer electricity. Each supplier submit a price function to the market, to which the market reacts by fixing the quantities bought to each supplier. The objective of the market is to satisfy its fixed demand, whenever possible, at minimal price.

We investigate two cases. In the first case, each of the suppliers strives to maximize its market share on the market; in the second case each supplier strives to maximize its profit.

We also make some remarks when suppliers may bring electricity on several markets.

Key-words: Game theory, Nash equilibrium, Spot markets, Electricity Market

This work was partially preformed while Geert Jan Olsder was on leave at CNRS/i3s/UNSA, Sophia Antipolis, France during the academic year 02/03

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Utilisation de la théorie des jeux pour l'étude des marchés d'électricité

Résumé : Nous étudions un modèle de jeu statique décrivant un marché spot d'électricité, où interagissent un certain nombre de fournisseurs. Chaque fournisseur soumet une fonction prix au marché. Le marché réagit à ces propositions en fixant les quantités d'électricité achetées à chaque fournisseur. L'objectif du marché est de satisfaire sa demande au meilleur prix.

Nous étudions successivement les cas oú les fournisseurs cherchent à maximiser leurs parts de marché et le cas où les fournisseurs cherchent à maximiser leurs profits.

Nous abordons brièvement le cas où les fournisseurs peuvent fournir de l'électricité sur plusieurs marchés concurrents.

Mots-clés : Théorie des jeux, Equilibre de Nash, Marchés spot, Marché d'électricité

1 Introduction

Since the deregulation process of electricity exchanges has been initiated in European countries, many different market structures have appeared (*see e.g. [11]*). Among them are the so called *day ahead markets* where suppliers face a decision process that relies on a centralized auction mechanism. It consists in submitting bids, more or less complicated, depending on the design of the *day ahead market* (power pools, power exchanges, ...). The problem is the determination of the quantity and price that will win the process of selection on the market. Our aim in this paper is to describe the behavior of the participants (suppliers) through a static game approach. We consider a market where $\mathcal{S}$ suppliers are involved. Each supplier offers on the market a price function. The response of the market to these offers is the quantities bought from each supplier. The objective of the market is to satisfy its demand at minimal price.

Closely related papers are [12] and [5]. They also consider optimal bids on electricity markets. Nevertheless, in [12], the authors take the quantity of electricity proposed on the market as exogenous, whereas here we consider the quantity as part of the bid. In [5], the authors do not consider exactly the same kind of market mechanism, in particular they consider open bids and fix the market clearing price as the highest price among the accepted bids. They consider fixed demand but also stochastic demand.

The paper is organized as follows. The model is described in Section 2, together with the proposed solution concept. In Section 3 we consider the case where the suppliers strive to maximize their market share, while in Section 4 we analyze the case where the goal is profit maximization. In Section 5 we introduce some alternative market organization that we compare with the organization studied in the previous sections. Some hints for the case where suppliers propose electricity on several markets are presented in Section 6. We finally conclude in Section 7 with some comparison remarks on the two criteria used, and some possible directions for future works.

2 Problem statement

2.1 The agents and their choices

We consider $\mathcal{M}$ electricity markets, called $M_1, \cdots, M_{\mathcal{M}}$ that have an inelastic (to price) demand for d_i units of electricity that is provided by $\mathcal{S}$ suppliers called S_j, $j = 1, 2, \ldots, \mathcal{S}$.

In most of the paper we will consider the case of a single market, ($\mathcal{M} = 1$). So we have chosen to describe here only this case. In Section 6 we will extend the subsection below to the case of several markets. Hence for now we consider the single market case.

2.1.1 The suppliers

Each supplier S_j sends an offer to the market that consists in a price function $p_j(\cdot)$, that associates to any quantity of electricity q, the unit price $p_j(q)$ at which it is ready to sell this quantity.

We shall mainly use the following special form of the price function :

Definition 1 *We consider the single market case. For supplier S_j, a quantity-price strategy, referred to as the pair (q_j, p_j), is a price function $p_j(\cdot)$ defined by*

$$p_j(q) = \begin{cases} p_j \geq 0, & \text{for } q \leq q_j, \\ +\infty, & \text{for } q > q_j. \end{cases} \tag{1}$$

q_j is the maximal quantity S_j offers to sell at the finite price p_j. For higher quantities the price becomes infinity.

Note that we use the same notation, (p_j) for the price function and for the fixed price. This should not cause any confusion.

Apart form the strategies previously defined, we will also make use of linear strategies in Section 5 and Section 6. More precisely a linear strategy, is such that the price for quantity q sold is given by

$$p(q) = \alpha q + \beta,$$

where α and β are positive parameters.

Note that the two sets of strategies, "quantity-price strategies" and "linear strategies" are disjoint.

2.1.2 The market

Suppose we have a single market. The market collects the offers made by the suppliers, *i.e.* the price functions $p_1(\cdot), p_2(\cdot), \cdots, p_{\mathcal{S}}(\cdot)$, and has to choose the quantities $\overline{q}_j$ to buy from each supplier S_j, $j = 1, \cdots, \mathcal{S}$. The unit price paid to S_j is $p_j(\overline{q}_j)$.
We suppose that an admissible choice of the market is such that the demand is fully satisfied at finite price, *i.e.* such that,

$$\sum_{j=1}^{\mathcal{S}} \overline{q}_j = d, \ \overline{q}_j \geq 0, \ \text{and } p_j(\overline{q}_j) < +\infty, \ \forall j. \tag{2}$$

When the set of admissible choices is empty, *i.e.* when the demand cannot be satisfied at finite cost (for example when the demand is too large with respect to some finite production capacity), then the market buys the maximal quantity of electricity it can at finite price, though the full demand is not satisfied.

2.2 Evaluation functions and objective

2.2.1 The market

We suppose that the objective of the market is to choose an admissible strategy (*i.e.* satisfying (2)), $(\overline{q}_1, \cdots, \overline{q}_{\mathcal{S}})$ in response to the offers $p_1(\cdot), \cdots, p_{\mathcal{S}}(\cdot)$ of the suppliers, so as to

minimize the total cost.
More precisely the market problem is :

$$\min_{\{\overline{q}_j\}_{j=1\cdots\mathcal{S}}} \varphi_M(p_1(\cdot),\cdots,p_{\mathcal{S}}(\cdot),\overline{q}_1,\cdots,\overline{q}_{\mathcal{S}}), \tag{3}$$

with

$$\varphi_M(p_1(\cdot),\cdots,p_{\mathcal{S}}(\cdot),\overline{q}_1,\cdots,\overline{q}_{\mathcal{S}}) \stackrel{\text{def}}{=} \sum_{j=1}^{\mathcal{S}} p_j(\overline{q}_j)\overline{q}_j, \tag{4}$$

subject to constraints (2).

Remark 1 *Note that the price at which the market buys electricity may be different from one supplier to the other.*
We could also study some market organization where, based upon the offers of the suppliers, the market set a unique price p^w at which it buys to any suppliers. The quantities bought to each supplier is determined according to the fixed price p^w and the offers of the suppliers. This market organization can be observed on markets such as the market Power Next in France or NordPool for nordic countries. This aspect will be briefly discussed in Section 5.

Remark 2 *Here we have omitted to take into account transportation cost. The underlying reason for that, is that we consider a single market and local producers. Hence we supposed that the transportation costs each supplier have to support are more or less the same.*
This assumption cannot be made in case of several markets (see Section 6).

2.2.2 The suppliers

The two criteria, profit and market share, will be studied for the suppliers :

- **The profit -** When the market buys quantities $\overline{q}_j,\ j=1,\cdots,\mathcal{S}$, supplier S_j's profit to be maximized is

$$\varphi_{S_j}(p_1(\cdot),\cdots,p_{\mathcal{S}}(\cdot),\overline{q}_j) \stackrel{\text{def}}{=} p_j(\overline{q}_j)\overline{q}_j - C_j(\overline{q}_j), \tag{5}$$

  where $C_j(\cdot)$ is the production cost function.

  Assumption 1 *We suppose that, for each supplier S_j, the production cost $C_j(\cdot)$ is a piecewise $\mathcal{C}^1$ and increasing convex function.*

  When C_j is not differentiable we define the marginal cost $C'_j(q)$ as
  $\lim_{\epsilon\to 0^+} \frac{dC_j}{dq}(q-\epsilon)$.
  Because of the assumption made on C_j, the marginal cost C'_j is monotonic and non-decreasing. In particular it can be a piecewise constant increasing function.

This assumption fits with the classical structure of marginal costs in electricity sector. As a matter of fact, marginal costs are increasing, piecewise constant : the producers starts producing in its cheapest production facility (whenever possible). If the market asks more electricity, the producer starts up the one but cheapest production facility *etc.* For a given production facility, the cost depends only on row material, and is linear with respect to production.

- **The market share -** for supplier S_j, $\overline{q}_j$ is the quantity bought from him by the market, *i.e.* we define this criterion as

$$\varphi_{S_j}(p_1(\cdot), \cdots, p_{\mathcal{S}}(\cdot), \overline{q}_j) \stackrel{\text{def}}{=} \overline{q}_j. \tag{6}$$

For this criterion, it is necessary to introduce a price constraint. As a matter of fact, the obvious, but unrealistic, solution without price constraint would be to set the price to zero whatever the quantity bought is.

We need a constraint such that, for example, the profit is non-negative, or such that the unit price is always above the marginal cost, C_j'.

Assumption 2 *For the sake of generality we suppose the existence of a minimal unit price function $\mathcal{L}_j$ for each supplier. Supplier S_j is not allowed to sell the quantity q at a unit price lower than $\mathcal{L}_j(q)$, i.e. for the market share criterion, an admissible price strategy should satisfy,*

$$q_j(q) \geq \mathcal{L}_j(q), \quad \forall q.$$

A natural choice for $\mathcal{L}_j$ is C_j', which expresses the usual constraint that the unit price is above the marginal cost. One could also choose $\mathcal{L}_j(q) = \frac{C_j(q)}{q}$ which expresses the fact that the profit cannot be negative.

2.3 Equilibria

From a game theoretical point of view, a two time step problem with $\mathcal{S}+1$ players will be formulated. At a first time step the suppliers announce their offers (the price functions) to the market, and at the second time step the market reacts to these offers by choosing the quantities $\overline{q}_j$ of electricity to buy from each supplier. Each player strives to optimize (*i.e.* maximize for the suppliers, minimize for the market) its own criterion function (φ_{S_j}, $j = 1, \cdots, \mathcal{S}$, φ_M) by properly choosing its own decision variable(s). The numerical outcome of each criterion function will in general depend on all decision variables involved. In contrast to conventional optimization problems, in which there is only one decision maker, and where the word "optimum" has an unambiguous meaning, the notion of "optimality" in games is open to discussion and must be defined properly. Various notions of "optimality" exist (see [1]).

Here the structure of the problem leads us to use a combined *Nash Stackelberg* equilibrium. Please note that the "leaders", *i.e.* suppliers, choose and announce functions $p_j(\cdot)$. In [1] the corresponding equilibrium is referred to as *inverse Stackelberg.*

More precisely, define $\{\overline{q}_j(p_1(\cdot), \cdots, p_{\mathcal{S}}(\cdot)), j = 1, \ldots, \mathcal{S}\}$, the best response of the market to the offers $(p_1(\cdot), \cdots, p_{\mathcal{S}}(\cdot))$ of the suppliers, *i.e.* a solution of the problem (2)-(3). The choices $\{p_j^*(\cdot), \overline{q}_j^*, \ \ j = 1, \cdots, \mathcal{S}\}$ will be said optimal if the following holds true,

$$\overline{q}_j^* = \overline{q}_j(p_1^*(\cdot), \cdots, p_{\mathcal{S}}^*(\cdot)), \tag{7}$$

For every supplier S_j, $j = 1, \ldots, \mathcal{S}$ and any admissible price function $\tilde{p}_j(\cdot)$ we have

$$\varphi_{S_j}(p_1^*(\cdot), \cdots, p_{\mathcal{S}}^*(\cdot), \overline{q}_j^*) \geq \varphi_{S_j}(p_1^*(\cdot), \cdots, \tilde{p}_j(\cdot), \cdots, p_{\mathcal{S}}^*(\cdot), \tilde{q}_j), \tag{8}$$

where

$$\tilde{q}_j \stackrel{\text{def}}{=} \overline{q}_j(p_1^*(\cdot), \cdots, \tilde{p}_j(\cdot), \cdots, p_{\mathcal{S}}^*(\cdot)). \tag{9}$$

The Nash equilibrium Equation (8) tells us that supplier S_j cannot increase its outcome by deviating unilaterally from its equilibrium choice $(p_j^*(\cdot))$. Note that in the second term of Equation (8), the action of the market is given by (9) : if S_j deviates from $p_j^*(\cdot)$ by offering the price function $\tilde{p}_j(\cdot)$, the market reacts by buying from S_j the quantity $\tilde{q}_j$ instead of $\overline{q}_j^*$.

Remark 3 *As already noticed the minimization problem (3)-(2) defining the behavior of the market may not have any solution. In that case the market reacts by buying the maximal quantity of electricity it can at finite price.*

At the other extreme, it may have infinitely many solutions (for example when several suppliers use the same price function). In that case $\overline{q}_j^*(\cdot)$ is not uniquely defined by Equation (7), consequently the Nash equilibrium defined by Equation (8) is not well defined.

We would need an additional rule that says how the market reacts when its minimization problem has several (possibly infinitely many) solutions. Such an additional rule could be, for example, that the market first buys from supplier S_1 then from supplier S_2, etc. or that the market prefers the offers with larger quantities, etc. Nevertheless, it is not necessary to make this additional rule explicit in this paper. So we do assume that there is an additional rule, known by all the suppliers that insures that the reaction of the market is unique.

3 Suppliers maximize market share

In this section we analyze the case where the suppliers strive to maximize their market shares by appropriately choosing the price functions $p_j(\cdot)$ at which they offer their electricity on the market. We restrict our attention to price functions $p_j(\cdot)$ given in Definition 1 and referred to as the *quantity-price* pair (q_j, p_j).

For supplier S_j we denote $\mathcal{L}_j(\cdot)$ its minimal unit price function that we suppose nondecreasing with respect to the quantity sold. Classically this minimal unit price function may represent the marginal production cost.

Using a *quantity-price* pair (q_j, p_j) for each supplier, the market problem (3) can be written as

$$\begin{array}{cc} & \min_{\{\overline{q}_j, j=1,\dots,\mathcal{S}\}} \sum_{j=1}^{\mathcal{S}} p_j \overline{q}_j, \\ \text{under} & \\ & 0 \leq \overline{q}_j \leq q_j, \ \sum_{j=1}^{\mathcal{S}} \overline{q}_j = d. \end{array} \tag{10}$$

To define a unique reaction of the market we use Remark 3, when Problem (10) does not have any solution (*i.e.* when $\sum_{j=1}^{\mathcal{S}} q_j < d$) or at the other extreme when Problem (10) has (possibly infinitely) many solutions.

Hence we can define the evaluation function of the suppliers by

$$J_{S_j}((q_1, p_1), \cdots, (q_{\mathcal{S}}, p_{\mathcal{S}})) = \overline{q}_j(p_1(\cdot), \cdots, p_{\mathcal{S}}(\cdot)),$$

where the price function $p_j(\cdot)$ is the pair (q_j, p_j) and $\overline{q}_j(p_1(\cdot), \cdots, p_{\mathcal{S}}(\cdot))$ is the unique optimal reaction of the market.

Now the Nash Stackelberg solution can be simply expressed as a Nash solution, *i.e.* find $\mathbf{u}^* \stackrel{\text{def}}{=} (u_1^*, \cdots, u_{\mathcal{S}}^*)$, $u_j^* \stackrel{\text{def}}{=} (q_j^*, p_j^*)$, so that for any supplier S_j and any pair $\tilde{u}_j = (\tilde{q}_j, \tilde{p}_j)$ we have

$$J_{S_j}(\mathbf{u}^*) \geq J_{S_j}(\mathbf{u}^*_{-j}, \tilde{u}_j), \tag{11}$$

where $(\mathbf{u}^*_{-j}, \tilde{u}_j)$ denotes the vector $(u_1^*, \cdots, u_{j-1}^*, \tilde{u}_j, u_{j+1}^*, \cdots, u_{\mathcal{S}}^*)$.

Assumption 3 *We suppose that there exist quantities Q_j for $j = 1, \dots, \mathcal{S}$, such that*

$$\mathcal{L}_j(Q_j) < +\infty \ \textit{and} \ \sum_{j=1}^{\mathcal{S}} Q_j \geq d, \tag{12}$$

The quantities Q_j represent the maximal quantities of electricity supplier S_j can offer to the market. It may reflect maximal production capacity for producers or more generally any other constraints such that transportation constraints.

Remark 4 *The condition (12) insures that shortage can be avoided even if this implies high, but finite, prices.*

We consider successively in the next subsections the cases where the minimal price functions $\mathcal{L}_j$ are strictly increasing and continuous (Subsection 3.1) or non decreasing (possibly constant on some interval), and discontinuous (Subsection 3.2). This last case is the most important from the application point of view, since we often take $\mathcal{L}_j = C'_j$ which is not in general continuous.

3.1 Strictly increasing minimal price functions

We suppose the following assumption holds,

Assumption 4 *For any supplier S_j, $j \in \{1, \cdots, \mathcal{S}\}$ the minimal price function $\mathcal{L}_j$ is strictly increasing from $[0, Q_j]$ to $\mathbb{R}^+$ and $\lim_{y\to x} \mathcal{L}_j(y) = \mathcal{L}_j(x)$, for all $x \in [0, Q_j]$.*

Proposition 1

1. *Suppose that Assumption 4 holds. Then any strategy profile $\mathbf{u}^* = (u_1^*, u_2^*, \cdots, u_{\mathcal{S}}^*)$ with $u_j^* = (q_j^*, p^*)$ such that*
$$\begin{array}{l} \mathcal{L}_j(q_j^*) = p^*, \ \ \forall j \in \{1, \cdots, \mathcal{S}\} \text{ such that } q_j^* > 0 \\ \mathcal{L}_j(0) \geq p^*, \ \ \forall j \in \{1, \cdots, \mathcal{S}\} \text{ such that } q_j^* = 0 \\ \sum_{j \in \{1,2\cdots \mathcal{S}\}} q_j^* = d, \end{array} \tag{13}$$
is a Nash equilibrium.

2. *Suppose furthermore that Assumption 3 holds, then the equilibrium exists and is unique.*

Proof
First note that because of the last equation in (13) the solution of the minimization problem of the market (10) is unique (the suppliers satisfy exactly the demand), and hence the values of the evaluation functions of the suppliers are well defined.

1. We show that none of the Supplier can increase its profit by unilaterally deviate from its strategy u_j^*. For that, it is sufficient to show that for any $u_j \neq u_j^*$, $u_j^* = (q_j^*, p^*)$, we have
$$J_{S_j}(\mathbf{u}_{-j}^*, u_j^*) \geq J_{S_j}(\mathbf{u}_{-j}^*, u_j),$$
First note that, because of the last equation in (13), the market buys the total quantity of electricity proposed by the suppliers, *i.e.* we have $\overline{q}_j = q_j^*$, and then $J_{S_j}(\mathbf{u}^*) = q_j^*$. We now examine the two possible types of deviation for supplier j, namely, S_j may choose a strategy $u_j = (q_j, p_j)$, such that the quantity q_j is either lower or strictly higher than q_j^* :
   - If u_j is such that $q_j \leq q_j^*$, then obviously the demand is not anymore satisfied and the market has to buy the total quantity q_j to S_j, hence we have $J_{S_j}(\mathbf{u}_{-j}^*, u_j) = q_j \leq q_j^* = J_{S_j}(\mathbf{u}^*)$.
   - If u_j is such that $q_j > q_j^*$, then because $\mathcal{L}_j$ is strictly increasing by Assumption 4 and because of the price constraint, we must have $p_j \geq \mathcal{L}_j(q_j) > \mathcal{L}_j(q_j^*) = p^*$. So after the deviation, Supplier S_j propose its electricity at a price higher than the price p^* of the other suppliers. Hence, from the market reaction (10), we deduce that the market first buy to the other suppliers, and then complete the demand to S_j, *i.e.* buys the quantity $d - \sum_{i\neq j} q_i^* = q_j^*$ to S_j. Consequently we have,
$$J_{S_j}(\mathbf{u}_{-j}^*, u_j) = q_j^* = J_{S_j}(\mathbf{u}^*).$$

2. From Assumption 3 and Assumption 4, the function $\mathcal{L}_j^{-1}$ is well defined from $[\mathcal{L}_j(0), +\infty]$ to $[0, Q_j]$. We consider the prolongation of this function (that we still denote $\mathcal{L}_j^{-1}$), by defining $\mathcal{L}_j^{-1}(p) = 0$ for any $0 \le p \le \mathcal{L}_j(0)$. We can now define a function O, the total offer, by
$$\begin{array}{rccc} O: & [0, +\infty] & \longrightarrow & [0, \sum_{j=1}^{\mathcal{S}} Q_j] \\ & p & \longrightarrow & \sum_{j=1}^{\mathcal{S}} \mathcal{L}_j^{-1}(p) \end{array}$$
O is a non decreasing continuous function taking finite values in the interval $[0, \sum_j Q_j[$. Since by Assumption 3 we have $\sum_j Q_j > d$, there exists a unique p^* that solves Problem (13), *i.e.* such that $O(p^*) = d$. The quantities q_j^* are defined as $q_j^* = \mathcal{L}_j^{-1}(p^*)$, and because of the extension of the definition of $\mathcal{L}_j^{-1}$, it is clear that $q_j^* = 0$, for j such that $\mathcal{L}_j(0) \ge p^*$.
This ends the proof.

$\diamond$

Remark 5 *Note that we have a condition for a new entrance on the market. As a matter of fact, when the market is at equilibrium, let p^* be the equilibrium price. If for a new supplier that tries to enter the market, its minimal price $\mathcal{L}_j(0) \ge p^*$, then it cannot win any market share on that market.*

3.2 Nondecreasing minimal price functions

We now address the problem where the minimal price functions $\mathcal{L}_j$ are not necessarily strictly increasing. Nevertheless we assume that they are nondecreasing. We set the following assumption,

Assumption 5 *We suppose that the minimal price functions $\mathcal{L}_j$ are nondecreasing, piecewise continuous, and that $\lim\limits_{y \to x^-} \mathcal{L}_j(y) = \mathcal{L}_j(x)$ for any $x \ge 0$.*

Replacing Assumption 4 by Assumption 5, there may not be any strategy profile (see Figure 2) or at the other extreme there may be possibly infinitely many strategy profiles (see Figure 1) that satisfy Equations (13). Proposition 1 fails to characterize the Nash equilibria.

For any $p \ge 0$, we define $\rho_j(p)$ as the right continuous inverse of $\mathcal{L}_j$, or with other words as the maximal quantity supplier S_j can offer at price p, *i.e.*

$$\rho_j(p) = \begin{cases} \max\{q \ge 0, \mathcal{L}_j(q) \le p\}, & \text{if } j \text{ is such that } \mathcal{L}_j(0) \le p, \\ 0, & \text{otherwise .} \end{cases} \tag{14}$$

Hence $\rho_j(p)$ is only determined by the structure of the minimal price function $\mathcal{L}_j$. In particular it is not dependent on any choice of the suppliers.

As a consequence of Assumption 5, $\rho_j(p)$ increases with p, and for any $p \ge 0$, $\lim_{y \to p^+} \rho_j(y) = \rho_j(p)$.

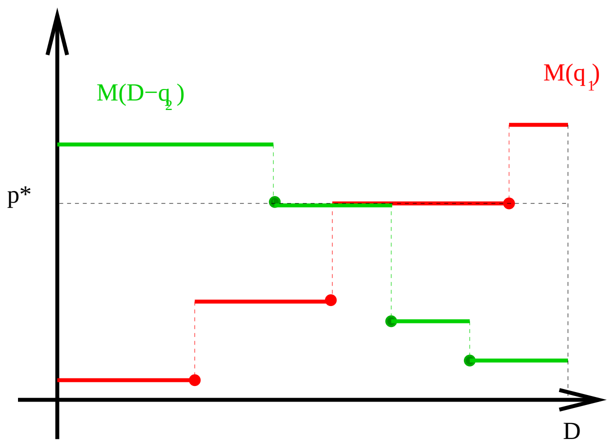

Figure 1: Non uniqueness

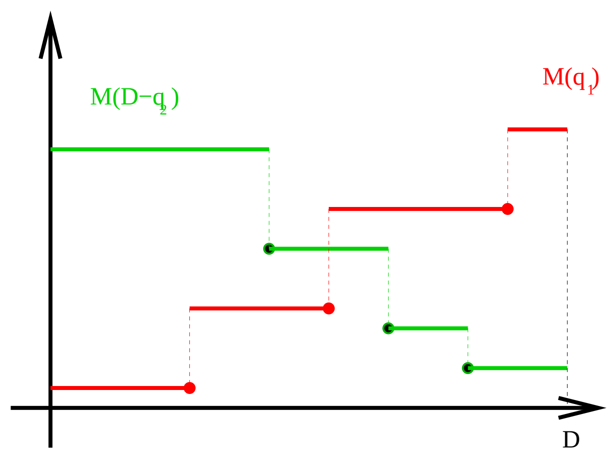

Figure 2: Non existence

Denote by $O(\cdot)$ the function from $\mathbb{R}^+$ to $\mathbb{R}^+$ defined by

$$O(p) = \sum_{j=1}^{\mathcal{S}} \rho_j(p). \tag{15}$$

$O(p)$ is the maximal total offer that can be achieved at price p by the suppliers respecting the price constraints. The function O is possibly discontinuous, nondecreasing (but not necessarily strictly increasing) and satisfies $\lim_{y \to p^+} O(y) = O(p)$. Assumption 3 implies that

$$O(\sup_j \mathcal{L}_j(Q_j)) \;\geq\; \sum_{j=1}^{\mathcal{S}} \rho_j(\mathcal{L}_j(Q_j)) \;\geq\; \sum_{j=1}^{\mathcal{S}} Q_j \;\geq\; d,$$

hence there exists a unique $p^* \leq \sup_j \mathcal{L}_j(Q_j) < +\infty$ such that

$$\begin{aligned} O(p^*) &= \sum_{j=1}^{\mathcal{S}} \rho_j(p^*) \geq d, \\ \forall \epsilon > 0, &\quad O(p^* - \epsilon) < d. \end{aligned} \tag{16}$$

The price p^* represents the minimal price at which the demand could be fully satisfied taking into account the minimal price constraint.

Assumption 6 *For p^* defined by (16), one of the following two condition holds,*

1. *There exists a unique $\bar{j} \in \{1, \cdots, \mathcal{S}\}$ such that $\mathcal{L}_{\bar{j}}^{-1}(p^*) \neq \emptyset$, where $\mathcal{L}_j^{-1}(p) \stackrel{\text{def}}{=} \{q \in [0,d], \quad \mathcal{L}_j(q) = p\}$. In particular, there exists a unique $\bar{j} \in \{1, \cdots, \mathcal{S}\}$ such that $\mathcal{L}_{\bar{j}}(\rho_{\bar{j}}(p^*)) = p^*$, and such that for $j \neq \bar{j}$, we have $\mathcal{L}_j(\rho_j(p^*)) < p^*$.*
2. *At price p^* the maximal total quantity suppliers are allowed to propose is exactly d, i.e. $\sum_{j=1}^{\mathcal{S}} \rho_j(p^*) = d$.*

Proposition 2 *Suppose Assumptions 5 and 6 hold. Consider the strategy profile $\mathbf{u}^* = (u_1^*, \cdots, u_{\mathcal{S}}^*)$, $u_j^* = (q_j^*, p_j^*)$ such that,*

- *p^* is defined by Equation (16),*
- *for $j \neq \bar{j}$, i.e. such that $\mathcal{L}_j(\rho_j(p^*)) < p^*$ (see Assumption 6), we have $q_j^* = \rho_j(p^*)$ and $p_j^* \in [\mathcal{L}_j(q_j^*), p^*[$,*
- *for $j = \bar{j}$, i.e. such that $\mathcal{L}_{\bar{j}}(\rho_{\bar{j}}(p^*)) = p^*$ (see Assumption 6), we have $q_{\bar{j}}^* \in [\min((d - \sum_{k \neq \bar{j}} q_k^*)\ ,\ \rho_{\bar{j}}(p^*))\ ,\ \rho_{\bar{j}}(p^*)]$, and $p_{\bar{j}}^* \in [p^*, \overline{p}[$, where $\overline{p}$ is defined by*

$$\overline{p} \stackrel{\text{def}}{=} \min\{\mathcal{L}_k(q_k^{*+}),\ k \neq \bar{j}\} \tag{17}$$

then, $\mathbf{u}^*$ is a Nash equilibrium.

Remark 6 *There exists an infinite number of strategy profiles that satisfy the conditions of Proposition 2 (the prices p_j^* are defined as elements of some intervals). Nevertheless, we can observe that there is no need for any coordination among the suppliers to get a Nash equilibrium. Each supplier can choose independently a strategy as described in Proposition 2, the resulting strategy profile is a Nash equilibrium. Note that this property does not hold in general for non-zero sum games (see the classical "battle of the sexes" game [4]). We can* *also observe that for each supplier the outcome is the same whatever the Nash equilibrium set. In that sense we can say that all these Nash equilibria are equivalent.*

A reasonable manner to select a particular Nash equilibrium is to suppose that the suppliers may strive for the maximization of their profits as an auxiliary criterion. More precisely, among the equilibria with market share maximization as criterion, they choose the equilibrium that brings them the maximal income. Because the equilibria we have found are independent, it is possible for each supplier to choose its preferred equilibrium. More precisely, with this auxiliary criterion, the equilibrium selected will be,

$$\begin{array}{l} q_j^* = \rho_j(p^*),\ p_j^* = p^* - \epsilon, \quad \textit{for } j \neq \bar{j} \ (\textit{i.e. such that } \mathcal{L}_j(\rho_j(p^*)) < p^*), \\ q_{\bar{j}}^* = \rho_{\bar{j}}(p^*),\ p_{\bar{j}}^* = \overline{p} - \epsilon, \end{array}$$

where ϵ can be defined as the smallest monetary unit.

Remark 7 *Assumption 6 is necessary for the solution of the market problem (10) to have a unique solution for the strategies described in Proposition 2, which are consequently well defined.*
If Assumption 6 does not hold, we would need to make the additional decision rule of the market explicit (see Remark 3). This is shown in the following example (Figure 3), with $\mathcal{S} = 2$. The Nash equilibrium may depend upon the additional decision rule of the market. In Figure 3, we have $\mathcal{L}_1(\rho_1(p^*)) = \mathcal{L}_2(\rho_2(p^*)) = p^*$ and $\rho_1(p^*) + \rho_2(p^*) > d$, where p^* is the

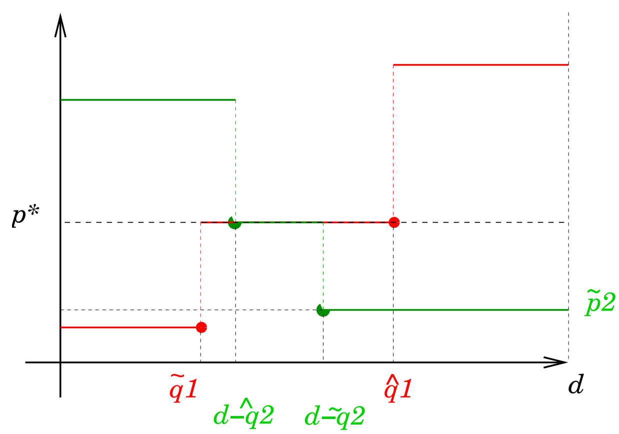

Figure 3: Example

price defined at (16). This means that Assumption 6 does not hold.

Suppose the additional decision rule of the market is to give the preference to supplier S_1, i.e. for a pair of strategies $((q_1, p), (q_2, p))$ such that $q_1 + q_2 > d$ the market reacts by buying the respective quantities q_1 and $d - q_1$ respectively to supplier S_1 and to supplier S_2.

The Nash equilibria for market share maximization are,

$$u_1^* = (q_1^* \in [d - \tilde{q}_2, \hat{q}_1], p^*), \quad u_2^* = (\tilde{q}_2, p_2^* \in [\tilde{p}_2, p^*[),$$

where $\hat{q}_i = \rho_i(p^*)$, $\tilde{q}_i = \rho_i(p^* - \epsilon)$, *and* $\tilde{p}_2 = \mathcal{L}_2(\tilde{q}_2)$.

Suppose now the additional decision rule of the market is a preference for supplier S_2. *The previous pair of strategies is not a Nash equilibrium any more. Indeed, supplier* S_2 *can increase its offer, at price* p^*, *to the quantity* $\hat{q}_2$. *The equilibrium in that case is*

$$u_1^* = (q_1^* \in [d - \hat{q}_2, \hat{q}_1], p^*), \quad u_2^* = (\hat{q}_2, p^*).$$

Remark 8 *In Proposition 2 we see that at equilibrium, the maximal price* $\bar{p}$ *that can be proposed is given by (17). A sufficient condition for that price to be finite is that for any* $j \in \{1, 2, \cdots, \mathcal{S}\}$ *we have,*

$$\sum_{k \neq j} Q_k > d. \tag{18}$$

Equation (18) means that with the withdrawal of an individual supplier, the demand can still be satisfied. This will insure that none of the suppliers can create a fictive shortage and then increase unlimitedly the price of electricity.

Proof of Proposition 2. We have to prove that for supplier S_j there is no profitable deviation of strategy, *i.e.* for any $u_j \neq u_j^*$, we have $J_{S_j}(\mathbf{u}_{-j}^*, u_j^*) \geq J_{S_j}(\mathbf{u}_{-j}^*, u_j)$.

- Suppose first that $j \notin \mathcal{S}(p^*)$ so that $\mathcal{L}_j(\rho_j(p^*)) < p^*$. Since for the proposed Nash strategy $u_j^* = (q_j^*, p_j^*)$, we have $p_j^* < p^*$, the total quantity proposed by S_j is bought by the market ($\overline{q}_j = q_j^*$). Hence $J_j(\mathbf{u}^*) = q_j^*$.
  - If the deviation $u_j = (q_j, p_j)$ is such that $q_j \leq q_j^*$, then clearly $J_{S_j}(\mathbf{u}^*) = q_j^* \geq q_j \geq J_{S_j}(\mathbf{u}_{-j}^*, u_j)$, whatever the price p_j is.
  - If the deviation $u_j = (q_j, p_j)$ is such that $q_j > \rho_j(p^*)$ then necessarily, by the minimal price constraint, Assumption 6 and the definition of $q_j^* = \rho_j(p^*)$, we have

    $$p_j \geq \mathcal{L}_j(q_j) \geq \mathcal{L}_j(q_j^{*\,+}) > \sup_{k \neq j} p_k^*.$$

    Hence now supplier S_j is the supplier with the highest price. Consequently the market first buys from the other suppliers and satisfies the demand, when necessary, with the electricity produced by supplier S_j (instead of the supplier $S_{\bar{j}}$). Hence the market share of S_j cannot increase with this deviation.

- Suppose now that $j = \bar{j}$, *i.e.* we have, $\mathcal{L}_{\bar{j}}(\rho_{\bar{j}}(p^*)) = p^*$.
  - If the first item of Assumption 6 holds, then at the proposed Nash equilibrium, supplier $S_{\bar{j}}$ is the supplier that meets the demand since it proposes the highest price.

Hence if supplier $S_{\bar{j}}$ wants to increases its market share, it has to sell a quantity $\tilde{q}_{\bar{j}} \geq d - \sum_{k \neq \bar{j}} q_k^*$. But we have,

$$\mathcal{L}_{\bar{j}}(\tilde{q}_{\bar{j}}) \geq \mathcal{L}_{\bar{j}}(d - \sum_{k \neq \bar{j}} q_k^*) = p^* > \max_{k \neq \bar{j}} p_k^*.$$

This proves that the quantity $\tilde{q}_{\bar{j}}$ cannot be offered at a price such that the market would buy it.

– If the second item of Assumption 6 holds, then the proposition states that the quantity proposed, and bought by the market is $\rho_{\bar{j}}$. An increase in the quantity proposed would imply an higher price, which would not imply an higher quantity proposed by the market since now the supplier would have the highest price.

$\diamond$

Now we suppose that Assumption 6 does not hold. So for the price p^* defined by (16) we have more than one supplier S_j such that $\mathcal{L}_j(\rho_j(p^*)) = p^*$.

As shown in the example of Remark 7 (see Figure 3), the Nash equilibria may depend upon the reaction of the market when two suppliers, S_i and S_j, have the same price $p_i = p_j = p^*$. It is clear that for a supplier S_j in such a way that $\mathcal{L}_j(\rho_j(p^*)) = p^*$, two possibilities may occur at equilibrium. Either, for some supplier S_j that fixes its price to $p_j = p^*$, the market reacts in such a way that $\overline{q}_j < \rho_j(p^* - \epsilon)$, in which case at equilibrium we will have $p_j^* = p^* - \epsilon$, or the market reacts such that $\overline{q}_j \geq \rho_j(p^* - \epsilon)$, and in that case we will have $p_j^* = p^*$.

Although the existence of Nash equilibria seems clear for any possible reaction of the market, we restrict our attention to the case where the market reacts by choosing quantities $(\overline{q}_j)_{j=1,\cdots\mathcal{S}}$ that are nondecreasing with respect to the quantity q_j proposed by each supplier S_j. More precisely we have the following assumption,

Assumption 7 *Let* $\mathbf{u} = (\mathbf{u_1}, \ldots, \mathbf{u}_{\mathcal{S}})$ *be a strategy profile of the suppliers with* $u_i = (q_i, p)$ *for* $i \in \{1, \cdots, \mathtt{k}\}$. *Suppose the market has to use its additional rule to decide how to share the quantity* $\tilde{d} \leq d$ *among suppliers* S_1 *to* $S_{\mathtt{k}}$ *(the quantity* $d - \tilde{d}$ *has already been bought from suppliers with a price lower than* p*).*

Let $i \in \{1, \cdots, \mathtt{k}\}$, *we define the function that associates* $\overline{q}_i$ *to any* $q_i \geq 0$, *where* $\overline{q}_i$ *is the* i*-th component of the reaction* $(\overline{q}_1, \cdots, \overline{q}_{\mathcal{S}})$ *of the market to the strategy profile* $(\mathbf{u}_{-i}, (q_i, p))$. *We suppose that this function is nondecreasing with respect to the quantity* q_i.

The meaning of this assumption is that the market does not penalize an "over-offer" of a supplier. For fixed strategies $\mathbf{u}_{-i}$ of all the suppliers but S_i, if supplier S_i, such that $p_i = p$ increases its quantity q_i, then the quantity bought by the market from S_i cannot decrease. It can increase or stay constant. In particular, it encompasses the case where the market has a preference order between the suppliers (for example, it first buys from supplier S_{j_1}, then supplier S_{j_2} etc...), or when the market buys some fixed proportion from each supplier. It does not encompass the case where the market prefers the smallest offer.

Proposition 3 *Suppose Assumption 6 does not hold while Assumption 7 does. Let the strategy profile* $((q_1^*, p_1^*), \cdots, (q_S^*, p_S^*))$ *be defined by*

- *For* j *such that* $\mathcal{L}_j(\rho_j(p^*)) < p^*$ *then,*

$$p_j^* = p^* - \epsilon, \quad q_j^* = \rho_j(p^* - \epsilon).$$

- *If* S_j *is such that* $\mathcal{L}_j(\rho_j(p^*)) = p^*$*, then either*

$$p_j^* = p^*, \quad q_j^* = \rho_j(p^*), \tag{19}$$

when the reaction of the market is such that

$$\overline{q}_j \geq \rho_j(p^* - \epsilon), \quad \forall \epsilon > 0,$$

or

$$p_j^* = p^* - \epsilon, \quad q_j^* = \rho_j(p^* - \epsilon). \tag{20}$$

when the new reaction of the market, for a deviation $p_j = p^*$ *would be such that* $\overline{q}_j < \rho_j(p^* - \epsilon)$.

This strategy profile is a Nash equilibrium.

Proof The proof follows directly from the discussion made before the proposition, and from the proof of Proposition 2. $\diamond$

Example We consider a market with 5 suppliers and a demand d equal to 10. We suppose that the minimal price functions $\mathcal{L}_j$ of suppliers are increasing staircase functions, given in the following table (the notation $(]a, b]; c)$ indicates that the value of the function in the interval $]a, b]$ is equal to c),

supplier 1	$([0,1];10), (]1,3];15), (]3,4];25), (]4,10];50)$
supplier 2	$([0,5];20), (]5,6];23), (]6,7];40), (]7,10];70)$
supplier 3	$([0,2];15), (]2,6];25), (]6,7];30), (]7,10];50)$
supplier 4	$([0,1];10), (]1,4];15), (]4,5];20), (]5,10];50)$
supplier 5	$([0,4];30), (]4,8];90), (]8,10];100)$

We display in the following table the values for $\rho_j(p)$ and $O(p)$ respectively defined by equations (14) and (15).

p	$\rho_1(p)$	$\rho_2(p)$	$\rho_3(p)$	$\rho_4(p)$	$\rho_5(p)$	$O(p)$
$p \in [0, 10[$	0	0	0	0	0	0
$p \in [10, 15[$	1	0	0	1	0	2
$p \in [15, 20[$	3	0	2	4	0	9
$p \in [20, 23[$	3	5	2	5	0	15

The previous table shows that for a price p in $[15, 20[$, only suppliers S_1, S_3 and S_4 can bring some positive quantity of electricity. The total maximal quantity that can be provided is 9 which is strictly lower than the demand $d = 10$. For a price in $[20, 23[$, we see that supplier S_2 can also bring some positive quantity of electricity, the total maximal quantity is then 15 which is higher than the demand. Then we conclude that the price p^* defined by equation (16) is $p^* = 20$. Moreover, $\mathcal{L}_2(\rho_2(p^*)) = \mathcal{L}_4(\rho_4(p^*)) = p^*$ which means that Assumption 6 is not satisfied. Notice that for supplier S_5, we have $\mathcal{L}_5(0) = 30 > p^*$. Supplier S_5 will not be able to sell anything to the market, hence, whatever its bid is, we have $\overline{q}_5 = 0$. We suppose that Assumption 7 holds. According to Proposition 3, we have the following equilibria.

$$u_1^* = (3, p_1^* \in [15, 20[), u_3^* = (2, p_3^* \in [15, 20[) \text{ and } u_5^* = (p^*, q^*),\ p^* \geq \mathcal{L}_5(q)$$

to which the market reacts by buying the respective quantities $\overline{q}_1(\mathbf{u}^*) = 3$, $\overline{q}_3(\mathbf{u}^*) = 2$ and $\overline{q}_5(\mathbf{u}^*) = 0$. The quantity 5 remains to be shared between S_2 and S_4 according to the additional rule of the market. For example, suppose that the market prefers S_2 to all other suppliers. Then

$$u_2^* = (q_2^* \in [1, 5], p_2^* = 20) \text{ and } u_4^* = (4, p_4^* \in [15, 20[).$$

to which the market reacts by buying $\overline{q}_2(\mathbf{u}^*) = 1$ and $\overline{q}_4(\mathbf{u}^*) = 4$. If now the market prefers S_4 to any other, then

$$u_2^* = (q_2^* \in [0, 5], p_2^* = 20) \text{ and } u_4^* = (5, p_4^* = 20).$$

to which the market reacts by buying $\overline{q}_2(\mathbf{u}^*) = 0$ and $\overline{q}_4(\mathbf{u}^*) = 5$.

4 Suppliers maximize profit

In this section, the objective of the suppliers is to maximize their profit, *i.e.* for a strategy profile $\mathbf{u} = (u_1, \cdots, u_{\mathcal{S}})$, $u_j = (q_j, p_j)$, their evaluation functions are

$$J_{S_j}(\mathbf{u}) = p_j \overline{q}_j - C_j(\overline{q}_j), \tag{21}$$

where $C_j(\cdot)$ denotes supplier S_j's production cost function, and $\overline{q}_j$ is the optimal reaction of the market, *i.e.* the solution of Problem (10) together with an additional decision rule, known by all the suppliers, in case of non-unique solutions (see Remark 3). As before, we do not need to make this rule explicit.

In contrast to the market share maximization, we do not need a minimal price functions $\mathcal{L}_j$. Nevertheless we need a maximal unit price $p_{\max}$ under which the suppliers are allowed to sell their electricity. This maximal price can either be finite and fixed by the market or be infinite.

From all the Assumptions previously made, we only retain in this section Assumption 1.

Lemma 1 *We define, for any finite price $p \geq 0$, $\widehat{Q}_j(p)$ as the set of quantities that maximizes the quantity $qp - C_j(q)$, i.e.*

$$\widehat{Q}_j(p) \stackrel{\text{def}}{=} \arg \max_{q \in [0,d]} qp - C_j(q), \tag{22}$$

and for infinite price,

$$\widehat{Q}_j(+\infty) = \min(Q_j, d), \tag{23}$$

where Q_j is the maximal production capacity of S_j.

We have for finite p,

$$\begin{array}{l} \widehat{Q}_j(p) = \{0\} \text{ if } C'(0) > p, \\ \widehat{Q}_j(p) = \{d\} \text{ if } C'(d) < p, \\ \widehat{Q}_j(p) = \{q, \ \ C'_j(q^-) \leq p \leq C'_j(q^+)\}, \text{ otherwise.} \end{array} \tag{24}$$

Proof We prove the last equality of (24). For any $q \in \widehat{Q}_j(p)$, we have for any $\epsilon > 0$,

$$pq - C_j(q) \geq p(q+\epsilon) - C_j(q+\epsilon),$$

from which we deduce that

$$\frac{C_j(q+\epsilon) - C_j(q)}{\epsilon} \geq p,$$

and letting ϵ tends to zero, it follows that $C'_j(q^+) \geq p$. The other equality is obtained with negative ϵ.
The first two equalities of (24) follow directly from the fact that C' is supposed to be nondecreasing. ◇

Note that if $C'(\cdot)$ is a continuous and nondecreasing function, then (24) is equivalent to the classical first order condition for the evaluation function of the supplier.

Lemma 2 *The function $p \rightarrow \max_{q \in [0,d]} (qp - C_i(q))$ is continuous and strictly increasing.*

Proof We recognize the Legendre-Fenchel transform of the convex function C_j. The continuity follows from classical properties of this transform.

The function is strictly increasing, since for $p > p'$, if we denote by $\tilde{q}$ a quantity in $\arg\max_{q \in [0,d]}(qp' - C_i(q))$, we have

$$\begin{aligned} \max_{q \in [0,d]} (qp - C_i(q)) & \geq \tilde{q}p - C_i(\tilde{q}) \\ & > \tilde{q}p' - C_i(\tilde{q}) = \max_{q \in [0,d]} (qp' - C_i(q)). \end{aligned}$$

◇

We now restrict our attention to the two suppliers' case, *i.e.* $\mathcal{S} = 2$.
Our aim is to determine the Nash equilibrium if such an equilibrium exists. Hence we need to find a pair $((q_1^*, p_1^*), (q_2^*, p_2^*))$ such that (q_1^*, p_1^*) is the best strategy of supplier S_1 if supplier

S_2 chooses (q_2^*, p_2^*), and conversely, (q_2^*, p_2^*) is the best strategy of supplier S_2 if supplier S_1 chooses (q_1^*, p_1^*). Equivalently we need to find a pair $((q_1^*, p_1^*), (q_2^*, p_2^*))$ such that there is no unilateral profitable deviation for any supplier $S_i,\ i = 1, 2$.

Let us determine the conditions which a pair $((q_1^*, p_1^*), (q_2^*, p_2^*))$ must satisfy in order to be a Nash equilibrium, *i.e.* no profitable deviation exists for any supplier. We will successively examine the case where we have an excess demand $(q_1^* + q_2^* \leq d)$ and the case where we have an excess supply $(q_1^* + q_2^* > d)$.

Excess demand : $q_1^* + q_2^* \leq d$ In that case the market buys all the quantities proposed by the suppliers, *i.e.* $\overline{q}_i^* = q_i^* \neq 0,\ i = 1, 2$.

1. Suppose that for at least one supplier, say supplier S_1, we have $p_1^* < p_{\max}$. Then supplier S_1 can increase its profit by increasing its price to $p_{\max}$. Since $q_1^* + q_2^* \leq d$ the reaction of the market to the new pair of strategies $((q_1^*, p_{\max}), (q_2^*, p_2^*))$ is still q_1^*, q_2^*. Hence the new profit of S_1 is now $q_1^* p_{\max} - C_1(q_1^*) > q_1^* p_1^* - C_1(q_1^*)$.
   We have exhibited a profitable deviation, $(q_1^*, p_{\max})$ for supplier S_1. This proves that a pair of strategies such that $q_1^* + q_2^* \leq d$ with at least one price $p_i^* < p_{\max}$ cannot be a Nash equilibrium.

2. Suppose that $p_1^* = p_2^* = p_{\max}$, and that there exists at least one supplier, say supplier S_1, such that $q_1^* = \overline{q}_1^* \notin \widehat{Q}_1(p_{\max})$, *i.e.* such that the reaction of the market does not maximize S_1's profit (see Lemma 1). Consequently, the profit for S_1, associated with the pair $((q_1^*, p_{\max}), (q_2^*, p_{\max}))$ is such that
   $$\overline{q}_1^* p_{\max} - C_1(\overline{q}_1^*) < \max_{q \in [0,d]} (q p_{\max} - C_1(q)).$$
   Since
   $$\lim_{\epsilon \to 0^+} \max_{q \in [0,d]} (q(p_{\max} - \epsilon) - C_1(q)) = \max_{q \in [0,d]} (q p_{\max} - C_1(q)),$$
   there exists some $\bar{\epsilon} > 0$ such that
   $$\max_{q \in [0,d]} (q(p_{\max} - \bar{\epsilon}) - C_1(q)) > \overline{q}_1^* p_{\max} - C_1(\overline{q}_1^*).$$
   This proves that any deviation $(\hat{q}_1, p_{\max} - \bar{\epsilon})$ of supplier S_1, such that $\hat{q}_1 \in \widehat{Q}_1(p_{\max} - \bar{\epsilon})$, is profitable for S_1.
   Hence, a pair of strategies such that $q_1^* + q_2^* \leq d$, $p_1^* = p_2^* = p^*$, to which the market reacts with, for at least one supplier, a quantity $\overline{q}_i^* \notin \hat{Q}_i(p_{\max})$ cannot be a Nash equilibrium.

3. Suppose that $p_1^* = p_2^* = p_{\max}$, $q_1^* = \overline{q}_1^* \in \hat{Q}_1(p_{\max})$ and $q_2^* = \overline{q}_2^* \in \hat{Q}_2(p_{\max})$ (*i.e.* the market reacts optimally for both suppliers).
   In that case the pair $((q_1^*, p_1^*), (q_2^*, p_2^*))$ is a Nash equilibrium. As a matter of fact no

deviation by changing the quantity can be profitable: since $\overline{q}_i^*$ is optimal for $p_{\max}$, the price cannot be increased, and a decrease of the profit will follow from a decrease of the price of one supplier (Lemma 2).

Excess supply : $q_1^* + q_2^* > d$ Two possibilities occur depending on whether the prices $p_j,\ j = 1, 2$ differ or not.

1. **the prices are different**, *i.e.* $p_1^* < p_2^*$ for example.
   In that case the market first buys from the supplier with lower price (hence $\overline{q}_1^* = \inf(q_1^*, d)$), and then completes its demand to the supplier with highest price, S_2 (hence $\overline{q}_2^* = d - \overline{q}_1^*$).
   For $\bar{\epsilon} > 0$ such that $p_1^* + \bar{\epsilon} < p_2^*$, we have
   $$q_1^* p_1^* - C_1(q_1^*) \leq \max_{q \in [0,d]} \{q p_1^* - C_1(q)\} < \max_{q \in [0,d]} \{q(p_1^* + \bar{\epsilon}) - C_1(q)\}.$$
   Hence supplier S_1 is better of increasing its price to $p_1^* + \bar{\epsilon}$ and proposing quantity $\hat{q}_1 \in \widehat{Q}_1(p_1^* + \bar{\epsilon})$. As a matter of fact, since $p_1^* + \bar{\epsilon} < p_2^*$, the reaction of the market will be $\overline{q}_1 = \hat{q}_1$.
   So a pair of strategies with $p_1 \neq p_2$ cannot be a Nash equilibrium.

2. **the prices are equal**, *i.e.* $p_1^* = p_2^* \stackrel{\text{def}}{=} p^*$.
   Now the market faces an optimization problem (10) with several solutions. Hence it has to use its additional rule in order to determine the quantities $\overline{q}_1^*, \overline{q}_2^*$ to buy from each supplier in response to their offers q_1^*, q_2^*.
   If this response is not optimal for any supplier, *i.e.* $\overline{q}_1^* \notin \widehat{Q}_1(p^*)$ and $\overline{q}_2^* \notin \widehat{Q}_2(p^*)$, the same line of reasoning as in Item 2 of the excess demand case proves that the pair $((q_1^*, p_1^*), (q_2^*, p_2^*))$ cannot be a Nash equilibrium.
   Suppose that the reaction of the market is optimal for both suppliers, *i.e.* $\overline{q}_1^* \in \widehat{Q}_1(p^*)$ and $\overline{q}_2^* \in \widehat{Q}_2(p^*)$. A necessary condition for a supplier, say S_1, to increase its profit is to increase its price and consequently to complete the offer of the other supplier S_2. We have two possibilities,

   (a) If for at least one supplier, say supplier S_1, we have,
   $$(d - q_2^*) p_{\max} - C_1(d - q_2^*) > \max_{q \in [0,d]} \{q p^* - C_1(q)\},$$
   then supplier S_1 is better of increasing its price to $p_{\max}$ and completing the market to sell the quantity $d - q_2^*$.

   (b) Conversely, if none of the suppliers can increase its profit by "completing the offer of the other", *i.e.* if

$$(d - q_2^*)p_{\max} - C_1(d - q_2^*) \le \max_{q \in [0,d]} \{qp^* - C_1(q)\}, \tag{25}$$

$$(d - q_1^*)p_{\max} - C_2(d - q_1^*) \le \max_{q \in [0,d]} \{qp^* - C_2(q)\}, \tag{26}$$

then the pair $((q_1^*, p_1^*), (q_2^*, p_2^*))$ is a Nash equilibrium.
As a matter of fact, for one supplier, say S_1, changing only the quantity is not profitable since $q_1^* \in \widehat{Q}_1$, decreasing the price is not profitable because of Lemma 2. Inequality (25) prevents S_1 from increasing its price.

Remark 9 *Note that a sufficient condition for Inequality (25) and (26) to be true, is that both suppliers choose $q_j^* = d$, $j = 1, 2$.*
With this choice each supplier prevents the other supplier from completing its demand with maximal price. This can be interpreted as a wish for the suppliers to obtain a Nash equilibrium. Nevertheless, to do that, the suppliers have to propose to the market a quantity d at price p^* which may be very risky and hence may not be credible. As a matter of fact, suppose S_1 chooses the strategy $q_1 = d, p_1 = p^* < p_{\max}$. If for some reason supplier S_2 proposes a quantity q_2 at a price $p_2 > p_1$, then S_1 has to provide the market with the quantity d at price p_1 since $\overline{q}_1 = d$, which may be disastrous for S_1.

Note also that if $p_{\max}$ is not very high compared with p^* the inequalities (25) and (26) will not be satisfied. Hence these inequalities could be used for the market to choose a maximal price $p_{\max}$ such that the equilibrium may be possible.

The previous discussion shows that in case of excess supply, the only possibility to have a Nash equilibrium, is that both suppliers propose the same price p^* and quantities q_1^*, q_2^*, such that, together with an additional rule, the market can choose optimal quantities $\overline{q}_1^*$, $\overline{q}_2^*$ that satisfy its demand and such that $\overline{q}_j^* \in \widehat{Q}_j(p^*)$.

This is clearly not possible for any price p^*. If the price p^* is too small, then the optimal quantity the suppliers can bring to the market is small, and for any $q \in \widehat{Q}_1(p^*) + \widehat{Q}_2(p^*)$, we have $q < d$. If the price p^* is too high, then the optimal quantity the suppliers are willing to bring to the market are large, and for any $q \in \widehat{Q}_1(p) + \widehat{Q}_2(p)$, we have $q > d$. The following Lemma characterizes the possible values of p^* for which it is possible to find q_1 and q_2 that satisfy $q_1 \in \widehat{Q}_1(p^*)$, $q_2 \in \widehat{Q}_2(p^*)$ and $q_1 + q_2 = d$.

Let us first define the function $\mathcal{C}'_j$ from $[0, d]$ to the set of intervals of $\mathbb{R}^+$ as

$$\mathcal{C}'_j(q) = [C'_j(q^-), C'_j(q^+)],$$

for q smaller than the maximal production quantity Q_j, and $\mathcal{C}_j(q) = \emptyset$ for $q > Q_j$. Clearly $\mathcal{C}'_j(q) = \{C'_j(q)\}$ except when C'_j has a discontinuity in q. We now can state the lemma.

Lemma 3 *It is possible to find q_1, q_2 such that $q_1 + q_2 = d$, $q_1 \in \widehat{Q}_1(p)$ and $q_2 \in \widehat{Q}_2(p)$ if and only if*

$$p \in \mathcal{I},$$

where,

$$\mathcal{I} \stackrel{\text{def}}{=} \bigcup_{q\in[0,d]} (\mathcal{C}_1'(q) \cap \mathcal{C}_2'(d-q)),$$

or, equivalently, when $Q_1 + Q_2 \geq d$, $\mathcal{I} \stackrel{\text{def}}{=} [\mathcal{I}^-, \mathcal{I}^+]$, *where*

$$\begin{aligned}\mathcal{I}^- &\stackrel{\text{def}}{=} \min\{p, \ \ \max(q \in \widehat{Q}_1(p)) + \max(q \in \widehat{Q}_2(p)) \geq d\},\\ \mathcal{I}^+ &\stackrel{\text{def}}{=} \max\{p, \ \ \min(q \in \widehat{Q}_1(p)) + \min(q \in \widehat{Q}_2(p)) \leq d\},\end{aligned}$$

and $\mathcal{I} = \emptyset$ *when* $Q_1 + Q_2 < d$.

Proof If $p \in \mathcal{I}$, then there exists $q \in [0,d]$ such that $p \in \mathcal{C}_1'(q)$ and $p \in \mathcal{C}_2'(d-q)$. We take $q_1 = q$, $q_2 = d - q$ and conclude by applying Lemma 1.
Conversely, if $p \notin \mathcal{I}$, then it is not possible to find q_1, q_2 such that $q_1 + q_2 = d$, and such that $p \in \mathcal{C}_1'(q_1) \cap \mathcal{C}_2'(q_2)$ *i.e.* such that according to Lemma 1 $q_1 \in \widehat{Q}_1(p)$ and $q_2 \in \widehat{Q}_2(p)$.
Straightforwardly, if $\mathcal{I}^- \leq p \leq \mathcal{I}^+$, then there exists $q_1 \in \widehat{Q}_1(p)$ and $q_2 \in \widehat{Q}_2(p)$ such that $q_1 + q_2 = d$. $\diamond$

We sum up the previous analysis by the following proposition.

Proposition 4 *In a market with maximal price* $p_{\max}$ *with two suppliers, each having to propose quantity and price to the market, and each one wanting to maximize its profit, we have the following Nash equilibrium:*

1. *If* $p_{\max} < \min\{p \in \mathcal{I}\}$-**Excess demand case**, *any strategy profile* $((q_1^*, p_{\max}), (q_2^*, p_{\max}))$, *with* $q_1^* \in \widehat{Q}_1(p_{\max})$ *and* $q_2^* \in \widehat{Q}_2(p_{\max})$ *is a Nash equilibrium. In that case we have* $q_1^* + q_2^* < d$.

2. *If* $p_{\max} = \min\{p \in \mathcal{I}\}$, *any pair* $((q_1^*, p_{\max}), (q_2^*, p_{\max}))$ *where* $q_1^* \in \widehat{Q}_1(p_{\max})$ *and* $q_2^* \in \widehat{Q}_2(p_{\max})$ *is a Nash equilibrium. In that case we may have* $q_1^* + q_2^* \geq d$ *or* $q_1^* + q_2^* < d$.

3. *If* $p_{\max} > \min\{p \in \mathcal{I}\}$- **Excess supply case**, *any pair* $((q_1^*, p^*), (q_2^*, p^*))$, *such that* $p^* \in \mathcal{I}$, $p^* \leq p_{\max}$ *and which induces a reaction* $(\overline{q}_1, \overline{q}_2)$, $\overline{q}_1 \leq q_1^*$, $\overline{q}_2 \leq q_2^*$, *such that*

   (a) $\overline{q}_1 + \overline{q}_2 = d$,

   (b) $\overline{q}_1 \in \widehat{Q}_1(p^*)$, $\overline{q}_2 \in \widehat{Q}_2(p^*)$,

   (c)
   $$\begin{aligned}(d - q_2^*)p_{\max} - C_1(d - q_2^*) &\leq \overline{q}_1 p^* - C_1(\overline{q}_1),\\ (d - q_1^*)p_{\max} - C_2(d - q_1^*) &\leq \overline{q}_2 p^* - C_2(\overline{q}_2),\end{aligned}$$

   is a Nash equilibrium.

We now want to generalize the previous result to a market with $\mathcal{S} \geq 2$ suppliers.

Let $((q_1^*, p_1^*), \ldots, (q_{\mathcal{S}}^*, p_{\mathcal{S}}^*))$ be a strategy profile, and let $(\overline{q}_1, \ldots, \overline{q}_{\mathcal{S}})$ be the induced reaction of the market. This strategy profile is a Nash equilibrium, if for any two suppliers, S_i, S_j, the pair of strategies $((q_i^*, p_i^*), (q_j^*, p_j^*))$ is a Nash equilibrium for a market with two suppliers (with evaluation function defined by Equation (21)) and demand $\tilde{d} = d - \sum_{k \notin \{i,j\}} \overline{q}_k$. Hence using the above Proposition 4, we know that necessarily at equilibrium the prices proposed by the suppliers are equal, and the quantities q_i^* induce a reaction of the market such that $\overline{q}_i \in \widehat{Q}_i(p^*)$.

Let us first extend the previous definition of the set $\mathcal{I}$ by $\mathcal{I} = \emptyset$ if $\sum_{j=1}^{\mathcal{S}} Q_j < d$, and otherwise,

$$\begin{array}{l} \mathcal{I} \stackrel{\text{def}}{=} [\mathcal{I}^-, \mathcal{I}^+], \\ \text{where} \\ \mathcal{I}^- = \min\{p, \ \sum_{j=1}^{\mathcal{S}} \max(q \in \widehat{Q}_j(p)) \geq d\}, \\ \mathcal{I}^+ = \max\{p, \ \sum_{j=1}^{\mathcal{S}} \min(q \in \widehat{Q}_j(p)) \leq d\}. \end{array} \tag{27}$$

We have the following

Theorem 1 *Suppose we have $\mathcal{S}$ suppliers on a market with maximal price $p_{\max}$ and demand d.*

1. *If $p_{\max} < \min\{p \in \mathcal{I}\}$-* **Excess demand case**, *any strategy profile $((q_1^*, p_{\max}), \cdots, (q_{\mathcal{S}}^*, p_{\max}))$, with $q_j^* \in \widehat{Q}_j(p_{\max})$, $j = 1, \cdots, \mathcal{S}$, is a Nash equilibrium. In that case we have $\sum_{j=1}^{\mathcal{S}} q_j^* < d$.*
2. *If $p_{\max} = \min\{p \in \mathcal{I}\}$, any strategy profile $((q_1^*, p_{\max}), \cdots, (q_{\mathcal{S}}^*, p_{\max}))$ where $q_j^* \in \widehat{Q}_j(p_{\max})$, $j = 1, \cdots, \mathcal{S}$ is a Nash equilibrium. In that case we may have $\sum_{j=1}^{\mathcal{S}} q_j^* \geq d$ or $\sum_{j=1}^{\mathcal{S}} q_j^* < d$.*
3. *If $p_{\max} > \min\{p \in \mathcal{I}\}$-* **Excess supply case**, *any strategy profile $((q_1^*, p^*), \cdots, (q_{\mathcal{S}}^*, p^*))$, such that $p^* \in \mathcal{I}$, $p^* \leq p_{\max}$ and which induces a reaction $(\overline{q}_1, \cdots, \overline{q}_{\mathcal{S}})$, $\overline{q}_j \leq q_j^*$, $j = 1, \cdots, \mathcal{S}$, such that*
   (a) $\sum_{j=1}^{\mathcal{S}} \overline{q}_j = d$,
   (b) $\overline{q}_j \in \widehat{Q}_j(p^*)$, $j = 1, \cdots, \mathcal{S}$,
   (c) for any $j = 1, \cdots, \mathcal{S}$
   $$(d - \sum_{k \neq j} q_k^*) p_{\max} - C_j(d - \sum_{k \neq j} q_k^*) \leq \overline{q}_j p^* - C_j(\overline{q}_j), \tag{28}$$
   is a Nash equilibrium.

The previous results show that a Nash equilibrium always exists for the case where the profit is used by the suppliers as an evaluation function. For convenience we have supposed the existence of a maximal price $p_{\max}$. On real markets we observe that, usually this maximal price is infinite, since most markets do not impose a maximal price on electricity. Hence the interesting case is the case where $p_{\max} > \min\{p \in \mathcal{I}\}$. The case with $p_{\max} \leq \min\{p \in \mathcal{I}\}$, can be interpreted as a monopolistic situation. The demand is so large compared with the maximal price that each supplier can behave as if it is alone on the market.

When $p_{\max}$ is large enough, Proposition 4 and Theorem 1 exhibit some conditions for a strategy profile to be a Nash equilibrium. We can make several remarks.

Remark 10 *Note that conditions (28) are satisfied for $q_j^* = d$. Hence we can conclude that, provided that the market reacts in such a way that $\overline{q}_j \in \widehat{Q}_j$, the strategy profile $((d, p^*), \cdots, (d, p^*))$ is a Nash equilibrium. Nevertheless, this equilibrium is not realistic. As a matter of fact, to implement this equilibrium, the suppliers have to propose to the market a quantity that is higher than the optimal quantity, and which possibly may lead to a negative profit (when $p_{\max}$ is very large). The second aspect that may appear unrealistic is the fact that the suppliers give up their power of decision. As a matter of fact, they announce high quantities, so that (28) is satisfied, and let the market choose the appropriate $\overline{q}_j$.*

A more realistic way for (28) to be satisfied, is that for any supplier $\overline{j}$, $\sum_{j \neq \overline{j}} \geq d$.

Example We consider the market, with demand $d = 10$ and maximal price $p_{\max} = +\infty$, with the five suppliers already described page 16. In order to be able to compare the equilibria for both criteria, market share and profit, we suppose that the marginal cost is equal to the minimal price function displayed in the table page 16, *i.e.* $C'_j = \mathcal{L}_j$.

The following table displays the quantities $o(p) \stackrel{\text{def}}{=} \sum_{j=1}^{5} \min\{q \in \widehat{Q}_j(p)\}$ and $O(p) \stackrel{\text{def}}{=} \sum_{j=1}^{5} \max\{q \in \widehat{Q}_j(p)\}$.

p	$\in [0, 10[$	$= 10$	$\in]10, 15[$	$= 15$	$\in]15, 20[$	$= 20$	$\in]20, 23[$
$o(p)$	0	0	2	2	9	9	15
$O(p)$	0	2	2	9	9	15	15

From the above table we deduce that $\mathcal{I} = \{20\}$, hence the only possible equilibrium price is $p^* = 20$. As a matter of fact, we have $O(20) = 15 > 10 = d$, and for any $p < 20$, $O(p) < 10 = d$, and $o(20) = 9 < 10 = d$, and for any $p > 20$, $o(p) > 10 = d$. Hence $p^* = 20 \in \mathcal{I}$ as defined by Equation (27).
Now concerning the quantities, the equilibrium depend upon the additional rule of the market. We suppose that the market chooses $\overline{q}_i \in \widehat{Q}_i, \ \forall i \in \{1, \cdots, 5\}$, and then to give preference to S_1, then to S_2, *etc.*
The equilibrium is $q_1^* \geq 3, \quad q_2^* \geq 1, \quad q_3^* \geq 2, \quad q_4^* \geq 4, \quad q_5^* \geq 0$.

The fact that the market wants to choose quantities $\overline{q}_i \in \widehat{Q}_i(20)$ implies that $\overline{q}_1 \in \widehat{Q}_1(20) = 3, \quad \overline{q}_2 \in \widehat{Q}_2(20) = [0, 5], \quad \overline{q}_3 \in \widehat{Q}_3(20) = 2, \quad \overline{q}_4 \in \widehat{Q}_4(20) = [4, 5], \quad \overline{q}_5 = 0,$

and the preference for S_2 compared to S_5 implies that $\overline{q}_1 = 3, \quad \overline{q}_2 = 1, \quad \overline{q}_3 = 2, \quad \overline{q}_4 = 4, \quad \overline{q}_5 = 0$.

If the preference would have been S_5 then S_4 then S_3 etc... the equilibrium would have been the same, but we would have had

$$\overline{q}_1 = 3, \quad \overline{q}_2 = 0, \quad \overline{q}_3 = 2, \quad \overline{q}_4 = 5, \quad \overline{q}_5 = 0.$$

5 Single price versus multiple price market organization

Our aim in this section is to analyze and compare a second possible organization. Previously we used an organization market, referred to as *multiple price market organization*, where the market possibly buys to each supplier at a different price.

In practice when the number of suppliers becomes large, *i.e.* when the markets become more fluid. In that case we observe that the markets determine a single price, based upon the offers of the suppliers. In this section we describe such an organization, referred to as *single price market organization*, and we compare it to the *multiple price market organization*.
Note also that when the market becomes more fluid, it is more likely that Condition (18) introduced in Remark 8 applies, *i.e.* that none of the suppliers can create a fictive shortage and then increase unlimitedly the price.

Note that in the following we will not restrict our attention to the *quantity-price* strategies as defined by Definition 1.

5.1 Single price market organization

We suppose that each supplier provides the market with a price function strategy $p_j(q)$. According to these offers, the market deduces a unique minimal price p^s (and quantities $\overline{q}_j$ to buy to each supplier), that solves the optimization problem

$$\begin{aligned} p^s = \min_{\{\overline{q}_j\}_{j=1,\cdots,\mathcal{S}}} & p, \\ \text{subject to} & \\ \overline{q}_j \geq 0&, \\ \text{for } \overline{q}_j > 0, \quad p_j(\overline{q}_j) \leq p < +\infty&, \\ \text{and } \sum_{j=1}^{\mathcal{S}} \overline{q}_j = d&. \end{aligned} \tag{29}$$

Remark 11 *Note that the minimization problem (29) may not have any solution. This would be the case if the demand d is too high compared to some maximal production quantities of the suppliers. In that case, as previously, we suppose that the market buys as electricity*

it can at some maximal given price.
At the other extreme the minimization problem may have several solutions. In that case we are once more faced to the necessity to introduce an additional decision rule to determine the exact quantities bought by the market to each supplier. See Remark 3.

Equivalently, we can use supply functions $q_j(p)$ instead of the price function $p_j(q)$. For a given price function $p_j(\cdot)$ of S_j, we define the corresponding supply function as the right continuous inverse function of $p_j(\cdot)$, *i.e.*

$$q_j(p) = \max\{q \geq 0,\ p_j(q) \leq p\},$$

and the problem (29) of the market becomes,

$$p^s = \min\{p \mid \sum_{j=1}^{\mathcal{S}} q_j(p) \geq d\}. \tag{30}$$

If the price function is strictly increasing, then obviously the supply function is its inverse function. In that case the inequality (30) becomes an equality, and we have $\overline{q}_j = q_j(p^s)$.

5.2 Comparing the two market organizations

Here we show that the two market organizations are not equivalent. More precisely, we are going to show on a simple example, that the quantities of electricity sold by the suppliers are different from one organization to the other one.
As a matter of fact, suppose we have two suppliers. We show that for the two given offers,

$$p_1(q) = \alpha_1 q + \beta_1 \text{ and } p_2(q) = \alpha_2 q + \beta_2, \quad \alpha_1 \geq 0,\ \alpha_2 \geq 0,$$

the reaction of the market is different, in terms of prices and quantities bought to each supplier.

For the two linear strategies $p_1(\cdot)$ and $p_2(\cdot)$ of the suppliers it is equivalent to consider their supply functions (the inverse functions of the price functions), namely,

$$q_1(p) = \frac{1}{\alpha_1}p - \frac{\beta_1}{\alpha_1} \text{ and } q_2(p) = \frac{1}{\alpha_2}p - \frac{\beta_2}{\alpha_2}.$$

Let us compute the reaction of the market for the two market organizations.

- **Single price market organization** The market reacts by solving problem (30), namely find p^s solution of the minimization problem,

$$p^s = \min_{\overline{q}_1, \overline{q}_2} p, \tag{31}$$

subject to

$$p_1(\overline{q}_1) + p_2(\overline{q}_2) \leq p, \quad \overline{q}_1 + \overline{q}_2 = d. \tag{32}$$

Suppose that d is sufficiently large ($p_1(0) < p_2(d)$ and $p_2(0) < p_1(d)$). Because we have chosen linear strategies (*i.e.* strictly increasing strategies), it is equivalent to find p^s such that,

$$q_1(p^s) + q_2(p^s) = d.$$

It comes that

$$p^s = \frac{\alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 d}{\alpha_1 + \alpha_2}.$$

We deduce that the quantities bought to each suppliers, $\overline{q}_j^s = q_j(p^s)$, are respectively

$$\overline{q}_1^s = \frac{\beta_2 - \beta_1 + \alpha_2 d}{\alpha_1 + \alpha_2}, \text{ and } \overline{q}_2^s = \frac{\beta_1 - \beta_2 + \alpha_1 d}{\alpha_1 + \alpha_2},$$

and the cost paid by the market to satisfy its demand is,

$$C^s = \frac{d(\alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 d)}{\alpha_2 + \alpha_1}.$$

- **Multiple price market organization.** The market reacts by fixing the quantities $\overline{q}_1$ and $\overline{q}_2$ that solve problem (3)(2), *i.e.*

$$\min_{\overline{q}_1,\overline{q}_2} p_1(\overline{q}_1)\overline{q}_1 + p_2(\overline{q}_2)\overline{q}_2.$$

It comes straightforwardly that, for a demand d large enough (see above conditions),

$$\overline{q}_1 = \overline{q}_1^s + \frac{\beta_1 - \beta_2}{2(\alpha_1 + \alpha_2)} \quad \text{and} \quad \overline{q}_2 = \overline{q}_2^s - \frac{\beta_1 - \beta_2}{2(\alpha_1 + \alpha_2)}$$

from which we deduce that the unit price for each suppliers are

$$\overline{p}_1 = p_1(\overline{q}_1) = p^s + \frac{\alpha_1(\beta_1 - \beta_2)}{2(\alpha_1 + \alpha_2)} \quad \text{and} \quad \overline{p}_2 = p_2(\overline{q}_2) = p^s - \frac{\alpha_2(\beta_1 - \beta_2)}{2(\alpha_1 + \alpha_2)}.$$

and the total cost incurred by the market to fulfill its demand is

$$\overline{C} = C^s - \frac{1}{4}\frac{(\beta_1 - \beta_2)^2}{\alpha_1 + \alpha_2}.$$

Not surprisingly (considering the degree of freedom of the market), we observe that, unless $\beta_1 = \beta_2$, the price paid by the market is lower when it is allowed to set different prices for different suppliers.

Remark 12 *Note that β_j represents the fixed part of the price. Hence when both producers have the same fixed part, the two organizations (single price or multiple price) are equivalent (same prices, same quantities). For some fixed values of α_1 and α_2 the difference of cost between the two organization increases as the difference of fixed part between the suppliers $(\beta_2 - \beta_1)$ increases.*
Nevertheless this property does not hold when there are more than two suppliers.

5.3 Nash equilibrium for the two market organizations ?

The aim in this subsection is to examine in which case the markets with the two described organizations can be stabilized with a Nash equilibrium, when the suppliers use supply functions as strategies. We consider the case of profit maximization.

5.3.1 Single price organization

Suppose that the market announces a price p "small", then it is clear that the suppliers react by setting the quantities $\widehat{Q}_j(p)$ that maximizes the profit of Supplier S_j in a monopolistic situation, *i.e.* the quantity given by the supply function defined by Equation (22) :

$$\widehat{Q}_j(p) = \max_{q\in[0,d]} pq - C_j(q).$$

Hence for p fixed, the vector $(\widehat{Q}_1(p), \cdots, \widehat{Q}_{\mathcal{S}}(p))$ is a Nash equilibrium (provided that p is small enough, *i.e.* is such that $\sum_j \widehat{Q}_j(p) \leq d$).

Now if the market set the price p^w defined by

$$p^w = \min\{p \quad s.t. \ \sum_{j=1}^{\mathcal{S}} \widehat{Q}_j(p) = d\}, \tag{33}$$

where d is the fixed demand, the vector of quantities $(\widehat{Q}_1(p^w), \cdots, \widehat{Q}_{\mathcal{S}}(p^w))$ is a Nash equilibrium and furthermore the demand is fully satisfied.
Note that these equilibrium have been studied in [5], and in particular some extension when (33) does not have any solution (due to some discontinuities for example).

From the above reasoning, one would be tempted to deduce that the supply functions $(\widehat{Q}_1(\cdot), \cdots, \widehat{Q}_{\mathcal{S}}(\cdot))$ is a Nash equilibrium for our problem. Nevertheless we show, by exhibiting a counter example, that this is wrong.

Remark 13 *Note that it is not a contradiction that $(\widehat{Q}_1(p^w), \cdots, \widehat{Q}_{\mathcal{S}}(p^w))$ is a Nash equilibrium for q^w fixed, and that $(\widehat{Q}_1(\cdot), \cdots, \widehat{Q}_{\mathcal{S}}(\cdot))$ is not an equilibrium. As a matter of fact the role, and information, of the market and the suppliers are different in the two situations. In the first situation, the suppliers react to, and are informed of the action of the market (they know p^w), while in the second situation the reverse occurs : the market reacts to, and is informed of the actions of the suppliers.*

Counter Example We suppose that we have 2 suppliers on the market, that incur a quadratic production cost, *i.e.*

$$C_j(q) = c_j q^2.$$

The offer of each supplier on the market is a supply function $q_j(\cdot)$ that gives for any price p the quantity of electricity Supplier S_j commits itself to provide on the market. The market price p^s resulting from the pair of offers $(q_1(\cdot), q_2(\cdot))$ is the minimal price such that the (inelastic) demand d is fully satisfied, *i.e.* such that $q_1(p^s) + q_2(p^s) = d$. The profit for Supplier S_j is then

$$J_{S_j}(q_1(\cdot), q_2(\cdot)) = p^s q_j(p^s) - c_j q_j(p^s)^2.$$

Now we examine the supply function that gives for any price p the production that maximizes the evaluation function of the supplier (Equation 22). More precisely, for each fixed price p, the optimal quantity to bring on the market for Supplier S_j is

$$\hat{Q}_j(p) = \arg\max_q pq - c_j q^2 = \frac{p}{2c_j}.$$

For the strategy profile $(\hat{Q}_1(\cdot), \hat{Q}_2(\cdot))$, the market price is p^w such that $\widehat{Q}_1(p^w) + \widehat{Q}_2(p^w) = d$, *i.e.*

$$p^w = \frac{2dc_1c_2}{c_1 + c_2}.$$

It follows that the quantities of electricity sold by suppliers are

$$q_1^w = \frac{dc_2}{c_1 + c_2} \text{ and } q_2^w = \frac{dc_1}{c_1 + c_2},$$

leading to the profits,

$$J_{S_1}^w = \frac{d^2c_2^2c_1}{(c_1 + c_2)^2} \text{ and } J_{S_2}^w = \frac{d^2c_1^2c_2}{(c_1 + c_2)^2}$$

We show now that the pair $(\widehat{Q}_1(\cdot), \widehat{Q}_2(\cdot))$ is not a Nash equilibrium in the class of supply functions. To that aim, suppose that Supplier S_1 sticks on its choice $\hat{Q}_1(\cdot)$. We show that there is a profitable deviation for Supplier S_2, *i.e.* another supply function, such that S_2 increases its profit.

First let us keep in mind that at clearing the sum of the quantities sold is exactly equal to the demand d. This situation can be seen as supplier S_2 completes the quantity of supplier S_1 to fulfill the demand.

For any p, denotes $\psi_2(p)$, the quantity

$$\psi_2(p) = p(d - \widehat{Q}_1(p)) - c_2(d - \widehat{Q}_1(p))^2.$$

$\psi_2(p)$ represents the profit of supplier S_2, if the price fixed by the market is p and supplier S_1 sticks on the strategy $\hat{q}_1(\cdot)$.

Developing $\psi_2(p)$ it comes that

$$\psi_2(p) = \frac{1}{4c_1^2}(2dc_1 - p)(2pc_1 - 2c_1c_2d + c_2p).$$

Note that $\psi_2(p^w) = J_{S_2}(\widehat{Q}_1(\cdot), \widehat{Q}_2(\cdot))$. If we evaluate in $p = p^w$ the derivative of $\psi_2(p)$ with respect to p, we obtain that,

$$\frac{d\psi_2}{dp}(p^w) = \frac{dc_1}{c_1 + c_2} > 0.$$

From that we deduce that Supplier S_2 would be better of deviating from the strategy $\hat{q}_2(\cdot)$ to induce an higher price. We can see that $\psi_2(\cdot)$ attains its maximum for

$$p^* = \frac{2dc_1(c_1 + c_2)}{2c_1 + c_2}.$$

Hence any supply function $q_2(\cdot)$ such that the solution of $\hat{q}_1(p) + q_2(p) = d$ gives $p = p^*$ constitutes a best response of Supplier S_2 to the supply function $\widehat{Q}_1(\cdot)$ of Supplier S_1. In particular, we can choose the constant supply function $q_2^*(\cdot) = d - \widehat{Q}_1(p^*)$. This obviously is a profitable deviation for Supplier S_2. As a matter of fact, the clearing price for the pair of strategies $(\widehat{Q}_1(\cdot), q_2^*(\cdot))$ is p^*, the quantities sold are respectively $\hat{q}_1(p^*)$ and $d - \hat{q}_1(p^*)$, and from the previous calculations, the profit of Supplier 2 is higher than if he would have sold the quantity $d - \hat{q}_1(p^w)$.

For example, if we set $d = 1$ and

$$C_1(q) = q^2, \text{ and } C_2(q) = 2q^2,$$

we obtain the optimal monopolistic quantities

$$\hat{Q}_1(p) = \frac{p}{2}, \quad \hat{Q}_2(p) = \frac{p}{4}.$$

If the suppliers use the price strategies defined by $q_j(p) = \widehat{Q}_j(p)$, the resulting price on the market will be p^s such that

$$\widehat{Q}_1(p^s) + \widehat{Q}_2(p^s) = d = 1,$$

that is such that $\frac{p}{2} + \frac{p}{4} = 1$ that is $p^s = \frac{4}{3}$, from which it follows that the quantities of electricity sold by suppliers are $q_1^s = \frac{2}{3}$ and $q_2^s = \frac{1}{3}$. This bring them the respective profits

$$J_{S_1}(\frac{2}{3}, \frac{1}{3}) = \frac{4}{9}, \text{ and } J_{S_2}(\frac{2}{3}, \frac{1}{3}) = \frac{2}{9}.$$

Now, suppose that Supplier S_1 sticks on its choice $\widehat{Q}_1(q) = \frac{p}{2}$, and suppose that S_2 uses the constant supply function $\tilde{q}_2(q) = \frac{1}{4}$.
Now the clearing price p^s is such that

$$\frac{p^s}{2} + \frac{1}{4} = 1,$$

that is $p^s = \frac{3}{2}$.
The quantities sold respectively by S_1 and S_2 are $\widehat{Q}_1(\frac{3}{2}) = \frac{3}{4}$ and $\tilde{q}_2(\frac{3}{2}) = \frac{1}{4}$, while their profits become

$$J_{S_1}(\widehat{Q}_1(\cdot), \tilde{q}_2(\cdot)) = \frac{9}{8} \text{ and } J_{S_2}(\widehat{Q}_1(\cdot), \tilde{q}_2(\cdot)) = \frac{2}{8} > \frac{2}{9} = J_{S_2}(\widehat{Q}_1(\cdot), \tilde{q}_2(\cdot)).$$

This shows that the pair $(\widehat{Q}_1(\cdot), \widehat{Q}_2(\cdot))$ is not a Nash equilibrium since by deviating from this pair producer S_2 increases its profit.

Linear strategies and Nash Equilibrium Here we restrict our attention to linear strategies and to quadratic production cost.

Remark 14 *Note that linear strategies are of particular interest. As a matter of fact, usually the markets determine the class of strategies they will accept from the suppliers. We observe that these piecewise strategies are often chosen by the markets.*

Suppose that each supplier S_j chooses the linear production function

$$q_j(p) = a_j p - b_j, \qquad a_j > 0, \quad b_j > 0,$$

so as to maximize the profit given by,

$$p^s \overline{q}_j - c_j \overline{q}_j^2,$$

where p^s is the price fixed by the market, and $\overline{q}_j$ the quantity bought by the market to Supplier S_j.

The market reacts to the pair $(q_1(\cdot), q_2(\cdot))$ or equivalently to $((a_1, b_1), (a_2, b_2))$ by fixing the clearing price to p^s such that $q_1(p^s) + q_2(p^s) = d$. It comes that

$$p^s = \frac{b_1 + b_2 + d}{a_1 + a_2}.$$

Consequently the profit for each supplier is (as a function of (a_1, b_1, a_2, b_2)),

$$p^s q_j(p^s) - c_j q_j(p^s)^2, \tag{34}$$

that is

$$J_{S_1}(a_1, b_1, a_2, b_2) = \frac{(b_1 + b_2 + d)(a_1 b_2 + a_1 d - b_1 a_2)}{(a_1 + a_2)^2} - \frac{c_1(a_1 b_2 + a_1 d - b_1 a_2)^2}{(a_1 + a_2)^2}, \tag{35}$$

and

$$J_{S_2}(a_1, b_1, a_2, b_2) = \frac{(b_1 + b_2 + d)(a_2 b_1 + a_2 d - b_2 a_1)}{(a_1 + a_2)^2} - \frac{c_2(a_2 b_1 + a_2 d - b_2 a_1)^2}{(a_1 + a_2)^2}, \tag{36}$$

For the pair of production functions $(q_1^*(\cdot), q_2^*(\cdot))$ or with a slight abuse of notation, for the pair $((a_1^*, b_1^*), (a_2^*, b_2^*))$ to be a Nash equilibrium, the first order conditions for maximization, must hold true, so for $j = 1, 2$, we must have

$$\frac{\partial J_{S_j}}{\partial a_j}(a_1^*, b_1^*, a_2^*, b_2^*) = 0, \text{ and } \frac{\partial J_{S_j}}{\partial b_j}(a_1^*, b_1^*, a_2^*, b_2^*) = 0. \tag{37}$$

This can be rewritten as

$$\begin{aligned} \frac{A(d+b_1^*+b_2^*)}{(a_1^*+a_2^*)^3} &= 0 \\ \frac{A}{(a_1^*+a_2^*)^2} &= 0 \\ \frac{B(d+b_1^*+b_2^*)}{(a_1^*+a_2^*)^3} &= 0 \\ \frac{B}{(a_1^*+a_2^*)^2} &= 0 \end{aligned}$$

where A and B are defined by,

$$A = 2b_1^*a_2 + 2b_1^*c_1a_2^{*2} - a_1^*b_2^* + a_2^*b_2^* + a_2^*d - a_1^*d - 2c_1da_2^*a_1^* - 2c_1b_2^*a_2^*a_1^*,$$
$$B = 2b_2^*a_1^* + 2b_2^*c_2a_1^{*2} - a_2^*b_1^* + a_1^*b_1^* + a_1^*d - a_2^*d - 2c_2da_1^*a_2^* - 2c_2b_1^*a_1^*a_2^*.$$

Taking into account that $a_j^* > 0, b_j^* > 0$ the system (37) can be rewritten as

$$A = 0 \text{ and } B = 0.$$

This gives the family of solutions, *i.e.* of possible Nash equilibrium, parametrized by $a_j > 0$ and $b_j > 0$

$$\begin{aligned} &a_1^* = a_1, \quad a_2^* = a_2, \\ &b_1^* = \frac{(1+2c_2a_1)(a_1+2c_1a_2a_1-a_2)d}{2c_2a_2a_1+2c_1a_2a_1+a_1+a_2}, \\ &b_2^* = \frac{(1+2c_1a_2)(a_2+2c_2a_1a_2-a_1)d}{2c_1a_1a_2+2c_2a_1a_2+a_2+a_1}. \end{aligned} \tag{38}$$

Nevertheless a closer look at the second order conditions for maximization shows that the Hessian of $J_{S_j}(a_1, b_1, a_1^*, b_1^*)$ evaluated at any points satisfying the first order conditions for Nash equilibrium,(38), has a negative determinant[1]. This proves that the points satisfying Equation (38) are not Nash equilibrium.

Also we have checked that there cannot be any Nash equilibrium at the boundary of the domain, *i.e.* such that one of a_1, a_2, b_1 or b_2 is null.

Hence we do not have Nash equilibrium for this game in the class of linear strategies.

Remark 15 *Note that the computation above does not take into account any upper bound for the price function such as an upper bound for a_i and a lower bound for b_i or a maximal price for some given quantity (for example d).*

As in the previous part (for quantity-price strategies) the introduction of some maximal price may help to stabilize the market.

[1]These computations has been done using Maple. Because of their lengths we do not display here the expressions obtained for the Hessian and its determinant.

5.3.2 Multiple price market organization

As previously, we consider linear strategies and suppose that the suppliers incur a quadratic cost.

Here it is more convenient to work with price functions, *i.e.* functions $p_j : q \to p_j(q) = \alpha_j q + \beta_j$, with $\alpha_j > 0$ and $\beta_j \geq 0$.

The market reacts to a pair $(p_1(\cdot), p_2(\cdot))$ of price functions, by solving an optimization problem, where $\overline{q}_1$ and $\overline{q}_2$ are the optimization variables, the total cost is the objective function with the constraint that the demand d must be fully satisfied, that is

$$\min_{\overline{q}_1 \geq 0} p_1 \overline{q}_1 + p_2(d - \overline{q}_1), \quad \overline{q}_2 = d - \overline{q}_1.$$

It comes that

$$\begin{aligned}
\overline{q}_1 &= \frac{1}{2}\frac{\beta_2 - \beta_1 + 2\alpha_2 d}{\alpha_1 + \alpha_2} \\
\overline{q}_2 &= \frac{1}{2}\frac{\beta_1 - \beta_2 + 2\alpha_1 d}{\alpha_1 + \alpha_2}
\end{aligned}$$

Substituting these values into the profit function and deriving with respect to the coefficient α_j, β_j we obtain the following set of first order conditions for the Nash equilibrium,

$$\begin{aligned}
&(\beta_1 - \beta_2 - 2\alpha_2 d)(\ (\beta_2 + 2\alpha_2 d)(\alpha_1 - \alpha_2) + \beta_1(3\alpha_2 + \alpha_1) - 2c_1(\alpha_1 + \alpha_2)\ ) = 0 \\
&\beta_1(\alpha_1 + 2\alpha_2) - \alpha_2(\beta_2 + 2\alpha_2 d) - c_1(\alpha_1 + \alpha_2)) = 0 \\
&(\beta_2 - \beta_1 - 2d\alpha_1)(\ (\beta_1 + 2\alpha_1 d)(\alpha_2 - \alpha_1) + \beta_2(3\alpha_1 + \alpha_2) - 2c_2(\alpha_1 + \alpha_2)\ ) = 0 \\
&\beta_2(2\alpha_1 + \alpha_2) - \alpha_1(\beta_1 + 2\alpha_1 d) - c_2(\alpha_1 + \alpha_2)) = 0
\end{aligned}$$

A closer look to this system shows that it has no solution. Hence with this market organization there is no Nash equilibrium.

Remark 16 *As previously the above calculation are done without any upper bound on the price. The introduction of some upper bound may lead to stabilization of the market.*

6 The multiple market case

In this last section we present a model in order to study the case of multiple markets. We show that considering more than one market leads to a model that is much more complicated, both from the modeling point of view and the calculation point of view. Concerning modeling, a game appears at the level of the markets and the structure of the strategies of the suppliers are more complex.

In subsection 6.1 we describe a model to handle the multiple market case. In the multiple market case constraints resulting from interconnection and line capacity decreases the degree of freedom of the suppliers : they cannot sent as much electricity as they wish on the markets. We did dot take into account these constraints in the model we have described. Nevertheless they may lead to a simplest model, where some markets may be, in first approximation, considered as remote.

In Section 6.3 we investigate in a more abstract context the possibility to use inverse Stackelberg equilibrium.

6.1 Several market game model

6.1.1 Players and strategies

Here we consider that the $\mathcal{S}$ suppliers S_j propose electricity on $\mathcal{M}$ different markets, M_i. We suppose that each market M_i has a demand d_i inelastic to price. The aim of each market is to fulfill its demand at minimal price.

Suppliers' strategy. A strategy for a supplier S_j consists of a vector of price functions $p_j(\cdot) = (p_{j1}(\cdot), \cdots, p_{j\mathcal{M}}(\cdot))$ ($p_{ji}(\cdot)$ represents the offer of supplier S_j on market M_i). The markets react by setting the quantities $(\overline{q}_{j1}, \cdots, \overline{q}_{j\mathcal{M}})$ bought to each supplier.

As for the price function we can think that for a supplier S_j, the offer on each market $p_{ji}(\cdot)$, may have as argument the vector $(\overline{q}_{1i}, \cdots, \overline{q}_{\mathcal{M}i})$. Nevertheless, this does not reflect the actual organization of real electricity markets. As a matter of fact, a market will not accept an offer conditioned by the quantities $\overline{q}_{ji}$ of electricity bought by the other markets. For convenience, we will nevertheless use these strategies later on (Section 6.2) to illustrate some aspects of the model.

A more realistic way to define a price function strategy in that case, is to suppose that each supplier S_j splits its maximal production capacity in maximal capacity on each market. More precisely, we suppose that the maximal capacity supplier S_j is ready to bring to the market M_i is Q_{ji}, such that

$$\sum_{i=1}^{\mathcal{M}} Q_{ji} \leq Q_j,$$

where Q_j represents its maximal production capacity.

A price function $p_{ji}(\cdot)$ for supplier S_j on market M_i will be such that

$$\forall q > Q_{ji}, \;\; p_{ji}(q) = +\infty.$$

In the case where the suppliers want to maximize their market share, we have to set some condition that prevents them to fix a price arbitrarily low (see Assumption 2 for the single market case. Note that for the multiple market case the price fixed by the suppliers must include some transportation costs denoted by δ_{ji} (the transportation cost per unit of electricity). We suppose that for a local producer the transportation cost in zero ($\delta_{ii} = 0$). Hence, the previous condition may be to prevent the suppliers,

- to sell below the marginal cost. In that case a price function has to satisfy,

$$\forall i = 1, \cdots, \mathcal{M}, \forall j = 1, \cdots, \mathcal{S} \ \ \forall q_{ji} \leq Q_{ji}, \quad \min_{i=1,\cdots,\mathcal{M}} p_{ji}(\overline{q}_{ji}) \geq C'(\sum_{i=1}^{\mathcal{M}} \overline{q}_{ji}) + \delta_{ji}.$$

- or to have a negative profit. In that case, the price function has to satisfy,

$$\forall i = 1, \cdots, \mathcal{M}, \ \ \forall q_{ji} \leq Q_{ji}, \ \ \sum_{i=1}^{\mathcal{M}} p_{ji}(\overline{q}_{ji})\overline{q}_{ji} \geq C_j(\sum_{i=1}^{\mathcal{M}} q_{ji}) + \sum_{i=1}^{\mathcal{M}} \delta_{ji} q_{ji}. \tag{39}$$

Remark 17 *We see that the determination by the suppliers of the quantities Q_{ji} is crucial and depends upon the guess of the reaction of the market. As a matter of fact if for some market M_i the quantity Q_{ji} is much larger than the quantity $\overline{q}_{ji}$ the market M_i will actually buy, then the quantity $Q_{ji} - \overline{q}_{ji}$ cannot be sold anywhere else.*

The underlying assumption for this kind of price function, is that the offers of the suppliers has to be done before any market has decided of the quantities to buy.

If when setting an offer on some market, a supplier already knows the reaction of some other market, then this has to be taken into account in the definition of a strategy.

Markets' strategy For the market a strategy is the choice of quantities bought to each suppliers in reaction to the offers made by the suppliers.

6.1.2 Evaluation function and objective

We suppose that the objective of the markets is to minimize the total price paid for satisfying its demand, when possible, that is

$$J_{M_i} \stackrel{\text{def}}{=} \sum_{j=1}^{\mathcal{S}} p_{ji}(\overline{q}_{ji})\overline{q}_{ji}, \tag{40}$$

We suppose that the Market M_i reacts to an offer (or strategy profile) $\mathbf{p} = (p_1(\cdot), \cdots, p_{\mathcal{S}}(\cdot))$, or more precisely to the vector $(p_{1i}(\cdot), \cdots, p_{\mathcal{S}i}(\cdot))$ of the suppliers so as to satisfy its demand d_i and to minimize the total cost J_{M_i} (given by (40)).
We can consider that each market has specific market organization such as the one studied previously (*i.e.* single price market organization and multiple price market organization). The problem to solve by each market depends upon its organization. For example, if M_i follows a multiple price market organization as described previously, its problem to solve will be

$$\min_{\{\overline{q}_{ji}\}_{j=1,\cdots,\mathcal{S}}} \sum_{j=1}^{\mathcal{S}} p_{ji}(\overline{q}_{ji})\overline{q}_{ji}, \tag{41}$$

subject to

$$\overline{q}_{ji} \geq 0 \text{ and } \sum_{j=1}^{\mathcal{S}} \overline{q}_{ji} = d_i \text{ and } p_{ji}(\overline{q}_{ji}) < +\infty. \tag{42}$$

In the case where the constraint (42) cannot be satisfied (for example when the demand d_i is too large), then Markets M_i reacts by buying the maximal quantity at finite price, and its demand is not fully satisfied.

As for the suppliers we can consider as previously the two criteria : market share maximization or profit maximization.

Remark 18 *As in the single market case, the solution of the minimization problems of the markets may not lead to a unique choice of quantities bought to each suppliers. As previously (see Remarks 3 and 7) the markets need additional rule to set a unique reaction to the offers of the suppliers. We suppose that this additional rule is known to the suppliers before they choose their price function.*
Hence, the game can be viewed as a three stage problem : at first stage, the markets choose their additional rule, at a second stage the suppliers, with the knowledge of this additional rule set their offers on the markets, and at the las stage the market react to the offers by setting unique quantities bought to each suppliers.
The problem can be now viewed as a two step Nash-Nash problem. As a matter of fact, the offers made by the suppliers at the second step depend upon the additional rule of the markets $M_1, \cdots, M_{\mathcal{M}}$. Hence using backward induction reasoning, when at stage one, the markets decide upon the additional rule, they have to take into account the expected reaction of the suppliers to these additional rules. Hence there is an interdependence between the markets when choosing the additional rule. We make the assumption that the markets behave non cooperatively, and hence the have to determine a Nash at this initial stage. Hence we have now set a two level Nash game.

6.2 Two level Nash equilibrium with linear price function

The aim of this subsection (and the following) is to illustrate the difficulties that may appear when studying a two stage Nash-Nash problem as set at Remark 18.

We illustrate Nash-Nash calculations for suppliers' linear strategies. For the sake of simplicity, we omit the cost line.

We suppose here that the unit price function for each supplier S_j is linear and has as argument the total amount sold by S_j, and furthermore that it does not depend on the market. More precisely, consider unit price characterized by two parameters each;

$$p_j(\overline{q}_{j1} + \overline{q}_{j2}) = \alpha_j(\overline{q}_{j1} + \overline{q}_{j2}) + \beta_j. \tag{43}$$

We suppose that the parameters α_j and β_j are positive. As previously noticed, these strategies are not realistic, since they have as argument the total amount of electricity sold, that

is in particular, the offers on each market is correlated to the reaction of the other market, which is not realistic. Nevertheless these strategies allows us to illustrate Nash-Nash calculations.
We consider linear unit production cost for each supplier (with C_j the coefficient of proportionality). Hence the evaluation function for Supplier S_j is given by,

$$J_{S_j} = p_j(\overline{q}_{j1} + \overline{q}_{j2}).(\overline{q}_{j1} + \overline{q}_{j2}) - C_j.(\overline{q}_{j1} + \overline{q}_{j2}).$$

For a pair of unit price functions $(p_1(\cdot), p_2(\cdot))$ as in (43) and the respective reactions $(\overline{q}_{11}, \overline{q}_{21})$ and $(\overline{q}_{12}, \overline{q}_{22})$ of Markets M_1 and M_2, Market M_1 faces the costs

$$(\alpha_1(\overline{q}_{11} + \overline{q}_{12}) + \beta_1)\overline{q}_{11} + (\alpha_2(\overline{q}_{21} + \overline{q}_{22}) + \beta_2 + \delta_{21})\overline{q}_{21},$$

and market M_2 the cost

$$(\alpha_1(\overline{q}_{11} + \overline{q}_{12}) + \beta_1 + \delta_{12})\overline{q}_{12} + (\alpha_2(\overline{q}_{21} + \overline{q}_{22}) + \beta_2)\overline{q}_{22},$$

where it is tacitly understood that the lines S_1 to M_1 and S_2 to M_2 do not have transportation costs. The quantities $\overline{q}_{ji}$ have to satisfy

$$\overline{q}_{11} + \overline{q}_{21} = d_1,\ \overline{q}_{12} + \overline{q}_{22} = d_2,$$

for the total demands to be satisfied.

We suppose that the suppliers first announce their unit price function to which the markets reacts so as to minimize their total cost. Hence the solution adopted here is a two level Nash equilibrium or a Stackelberg Nash equilibrium. More precisely, at the first level (in time) the Suppliers (leaders) have to determine a Nash equilibrium taking into account the reaction of the markets, at the second level, given the strategies of the suppliers, the markets (followers) also have to determine a Nash equilibrium.

We use backward induction and first compute the Nash equilibrium at the follower level, *i.e.* the Nash equilibrium at the level of the market, given a pair of strategies of the Suppliers (the leaders).

Given a pair of strategies of the suppliers, *i.e.* a pair $((\alpha_1, \beta_1), (\alpha_2, \beta_2))$, the Nash equilibrium for the two markets is easily calculated to be

$$\begin{aligned}
\overline{q}_{11} &= \frac{-\beta_1 + \beta_2 + 3\alpha_2 d_1 + 2\delta_{21} + \delta_{12}}{3(\alpha_1 + \alpha_2)}, \\
\overline{q}_{21} &= \frac{\beta_1 - \beta_2 + 3\alpha_1 d_1 - 2\delta_{21} - \delta_{12}}{3(\alpha_1 + \alpha_2)}, \\
\overline{q}_{12} &= \frac{-\beta_1 + \beta_2 + 3\alpha_2 d_2 - 2\delta_{12} - \delta_{21}}{3(\alpha_1 + \alpha_2)}, \\
\overline{q}_{22} &= \frac{\beta_1 - \beta_2 + 3\alpha_1 d_2 + 2\delta_{12} + \delta_{21}}{3(\alpha_1 + \alpha_2)},
\end{aligned}$$

provided that all quantities concerned lie in the interior of their intervals and have the appropriate signs. Please note that these assumptions may not be valid in many cases (such is $\alpha_1 + \alpha_2 = 0$ excluded for instance). Also, the second-order conditions (players minimize, *i.e.* second derivative must be positive) must be satisfied. Though not derived explicitly, these second-order conditions are $\alpha_1 + \alpha_2 > 0$.

Remark 19 *Note that the quantities β_1 and β_2 appear only in a combined way as $\beta_1 - \beta_2$. This means that if a Nash solution is found for β_1^* and β_2^* say, then other Nash solutions exist for $\beta_1^* + \gamma$ and $\beta_2^* + \gamma$ with γ arbitrary. In other words, the Nash solution is not unique and the two suppliers will probably end up choosing their β_1 and β_2 arbitrarily high.*

The total production for S_1, S_2, respectively is

$$\overline{q}_{11} + \overline{q}_{12} = \frac{-2\beta_1 + 2\beta_2 + 3\alpha_2.(d_1 + d_2) + \delta_{22} - \delta_{12}}{3(\alpha_1 + \alpha_2)},$$

$$\overline{q}_{21} + \overline{q}_{22} = \frac{2\beta_1 - 2\beta_2 + 3\alpha_1.(d_1 + d_2) + \delta_{12} - \delta_{21}}{3(\alpha_1 + \alpha_2)}.$$

Short-hand notations for these two equations (to be used in a minute) are

$$x = \frac{n_1}{3d},\ y = \frac{n_2}{3d}.$$

What remains is to investigate the roles of the suppliers. The profit, with linear production cost, for the suppliers are

$$(\alpha_1(\overline{q}_{11} + \overline{q}_{12}) + (\beta_1 - C_1))(\overline{q}_{11} + \overline{q}_{12}), \quad (\alpha_2.(\overline{q}_{21} + \overline{q}_{22}) + (\beta_2 - C_2))(\overline{q}_{21} + \overline{q}_{22}),$$

respectively, in which the above equations for $\overline{q}_{11} + \overline{q}_{12}$ and $\overline{q}_{21} + \overline{q}_{22}$ must be substituted. With the notation just introduced, these profits can be written as

$$\alpha_1 x^2 + (\beta_1 - C_1)x, \quad \alpha_2 y^2 + (\beta_2 - C_2)y.$$

The first expression must be maximized with respect to β_1 and α_1 and the second one with respect to β_2 and α_2. Thus one obtains again a Nash equilibrium, but now on the level of the leaders (the suppliers). This leads to

$$x^2 + 2\alpha_1 x\frac{\partial x}{\partial \alpha_1} + (\beta_1 - C_1)\frac{\partial x}{\partial \alpha_1} = 0, \quad 2\alpha_1 x\frac{\partial x}{\partial \beta_1} + x + (\beta_1 - C_1)\frac{\partial x}{\partial \beta_1} = 0.$$

and

$$y^2 + 2\alpha_2 y\frac{\partial y}{\partial \alpha_2} + (\beta_2 - C_2)\frac{\partial y}{\partial \alpha_2} = 0, \quad 2\alpha_2 y\frac{\partial y}{\partial \beta_2} + y + (\beta_2 - C_2)\frac{\partial y}{\partial \beta_2} = 0.$$

Since

$$\frac{\partial x}{\partial \alpha_1} = \frac{n_1}{-3d^2}, \quad \frac{\partial x}{\partial \beta_1} = \frac{-2}{3d}, \quad \frac{\partial y}{\partial \alpha_2} = \frac{n_2}{-3d^2}, \quad \frac{\partial y}{\partial \beta_2} = \frac{-2}{3d},$$

one obtains

$$\begin{aligned}
\frac{n_1((n_1-3\beta_1)d-2\alpha_1 n_1)}{9d^3} &= 0,\\
\frac{-4\alpha_1 n_1+3d(n_1-2\beta_1)}{9d^2} &= 0,\\
\frac{n_2((n_2-3\beta_2)d-2\alpha_2 n_2)}{9d^3} &= 0,\\
\frac{-4\alpha_2 n_2+3d(n_2-2\beta_2)}{9d^2} &= 0,
\end{aligned}$$

in which derivation it has been assumed that $d=\alpha_1+\alpha_2\neq 0$. Solving these equations (actually, setting all numerators equal to zero), however, always leads to $d=0$. So, apparently the Nash solution does not exist, since we required that $\alpha_i>0$.

6.3 The inverse Stackelberg solution for the two two

In this section there are two suppliers S_1 and S_2 who announce their decisions u_{P_j} as a function of the quantities $u_{M_i} \overset{\mathrm{def}}{=} \overline{q}_{ji}$ that the markets M_1 and M_2 will buy. These functions will be indicated as $u_{P_j}=\gamma_j(u_{M_1},u_{M_2}),\ j=1,2$. Thus an inverse Stackelberg problem has been formulated. The cost functions, to be minimized, will be formally written as

$$J_{M_1}(u_{M_1},u_{P_1},u_{P_2}),\ J_{M_2}(u_{M_2},u_{P_1},u_{P_2}),\ J_{S_1}(u_{M_1},u_{M_2},u_{P_1}),\ J_{S_2}(u_{M_1},u_{M_2},u_{P_2}). \quad (44)$$

We will emphasize the particular structure of these functions rather than their proximity to reality. The interpretation of $J_{M_1}(u_{M_1},u_{P_1},u_{P_2})$ for instance is that the value of the criterion is determined by the total amount of electricity requested and this will depend on the price setting of the two suppliers, $u_{P_j}=\gamma_j(u_{M_1},u_{M_2}),\ j=1,2$. In practice, the market will buy an amount u_{1_i} from supplier S_1, with $u_{1_3}+u_{1_4}=u_{M_1}$, such as to get the electricity at the lowest price. The cost function J_1 does not directly depend on the amount of electricity required by the second market. For $J_{M_2}(u_{M_2},u_{P_1},u_{P_2})$ a similar explanation can be given. The cost function of the first supplier, i.e. S_1, depends on the total amount of electricity sold, i.e. on u_{M_1} and u_{M_2}, its own price setting, but not directly on the price setting of the other supplier. A similar remark explains the arguments of $J_{S_2}(u_{M_1},u_{M_2},u_{P_2})$.

In the remainder of this section, we will consider two subsections, one with one supplier and two markets, and the other one with one market and two suppliers. No realistic cost functions are envisaged. Because of this, we will talk about leaders rather than suppliers and about followers rather than markets. Only very limited results are known. for sake of simplicity, we will confine ourselves to cost functions as introduced above with scalar decision variables only.

6.3.1 One leader, two followers

The cost functions are symbolically given by

$$J_{M_1}(u_{M_1},u_{P_1}),\ J_{M_2}(u_{M_2},u_{P_1}),\ J_{S_1}(u_{M_1},u_{M_2},u_{P_1}).$$

Suppose that the absolute minimum of J_{S_1} is given by $u^*_{M_1}, u^*_{M_2}, u^*_{P_1}$. This solution will be achieved if all three players would help minimize J_{S_1}. The leader will try to obtain this team solution by enforcing the two followers play in such a way that while minimizing their own criteria, they simultaneously help the leader obtain his team minimum. Let us suppose that this is possible with a linear function

$$u_{P_1} = \gamma_1(u_{M_1}, u_{M_2}) = \alpha_1 u_{M_1} + \alpha_2 u_{M_2} + \alpha_3.$$

If possible, the coefficients α_i must satisfy (necessary conditions)

$$\begin{aligned} u^*_{P_1} &= \alpha_1 u^*_{M_1} + \alpha_2 u^*_{M_2} + \alpha_3, \\ \frac{\partial J_{M_i}(u^*_{M_i}, \gamma_1(u^*_{M_1}, u^*_{M_2}))}{\partial u_{M_i}} &= 0, \ i = 1, 2. \end{aligned}$$

Thus we have three equations with three unknowns. In [9] it is shown that this system of equations may be solvable and that the solution thus obtained satisfies the second order sufficiency conditions. This approach is directly extendibles to one supplier and many (more than two) markets.

6.3.2 Two leaders, one follower

The cost functions now have the following dependence

$$J_{M_1}(u_{M_1}, u_{P_1}, u_{P_2}), \ J_{S_1}(u_{M_1}, u_{P_1}), \ J_{S_2}(u_{M_1}, u_{P_2}).$$

One makes the following **hypothesis**. If the γ function of one leader is supposed to be given, then for the other leader a two person game remains with this leader and the follower as players. By properly choosing its own γ function, the latter leader can obtain his team minimum (*i.e.* as if all three players helped this particular leader in obtaining his absolute minimum). This hypothesis is known to generically hold when there are no constraints on the decision variables.

Now, if the leaders would react alternately to one another, alternately the solution to the problem would coincide with the absolute minimum of one leader and with the absolute minimum of the other leader. Obviously this process never converges and a Nash solution between the two leader does not exist.

If, however, constraints exist on the decision variables, then it is known that generically the team optimum cannot be obtained by either leader. Examples are known for this situation [9] in which the Nash equilibrium between the two leaders does now exist with the surprising property that it coincides with the team minimum of the follower!

7 Conclusion

We have shown in the previous sections that for both criteria, market share and profit maximization, it is possible to find a Nash equilibrium for a number $\mathcal{S}$ of suppliers. It is noticeable that for both cases the equilibrium price involved is the same (*i.e.* p^* given by Equation (15) for market share maximization and $p^* \in \mathcal{I}$ defined by Equation (27) for profit maximization), only the quantities proposed by the suppliers differ.

Nevertheless as already discussed in Remark 10, for profit maximization, the equilibrium strategies involved are not realistic in the interesting cases ($p_{\max}$ large). This may suggest that on these unregulated markets where suppliers are interested in instantaneous profit maximization, an equilibrium never occurs. Prices may becomes arbitrarily high and anticipation of the market behavior, and particularly market price, basically impossible.

This paper contains some modeling aspects that could be considered in more detail in future works. A first extension would be to consider more general suppliers. As a matter of fact, in the current paper, the evaluation functions chosen are more suitable for providers. Indeed, for profit maximization, we assumed that we had a production cost only for that part of electricity which is actually sold. This would fit the case where suppliers are producers. They produce only the electricity sold. The evaluation function chosen does not fit the case of traders who may have already bought some electricity and try to sell at best price the maximal electricity they have. The extension of our results in that case should not be difficult.

We supposed that every supplier perfectly knows the evaluation function of the other suppliers, and in particular their marginal costs. In general this is not true. Hence some imperfect information version of the model should probably be investigated.

Also the multiple market case which is definitely important should be investigated in more details. In particular, the impact of the constraints due to communication lines and interconnection between markets should be studied. Appreciating to what extend considering the markets as remote could be a reasonable approximation, would help the analysis in this domain.

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http://www.inria.fr
ISSN 0249-6399